Basic Notation

A set is a collection of objects. Order does not matter, only membership in the set. We use upper case letters, such as $A$, $B$, or $S$ to denote sets. Subscripts, such as $A_1$, $A_2$, or $A_3$ are also used.

The objects in a set are called the elements of the set or the members of the set. We typically use lower case letters, such as $a$, $b$, $x$ or $y$ to denote set elements. Set members are often subscripted, such as $a_1$ or $a_i$. Set membership is denoted by the $\in$ operator. The expression

$$x \in A$$

states that $x$ is a member of set $A$. The expression

$$x \notin A$$

states that $x$ is not a member of set $A$.

Sets are denoted by using a curly bracket pair to contain the list of set elements. For example,

$$\{1, 2, 4, 8\}$$

is the 4 element set whose members are the integers 1, 2, 4, and 8. An infinite set has an infinite number of elements in it. To denote infinite sets, or finite sets with a large number of elements, we can rely on 2 other forms of notation. If the set elements have a clear pattern to them, we can use an ellipsis to denote the set. The set

$$\{1, 2, 3, \ldots\}$$

is the set of natural numbers, or positive integers, often denoted by $\mathcal{N}$. The set of integers is often called $\mathcal{Z}$. The set of real numbers is $\mathcal{R}$ and the set of complex numbers is $\mathcal{C}$.

Set-builder notation can also be used to denote a set. This notation consists of a two part expression that describes the structure of each set member, and a condition that must be satisfied by each set member. Using set-builder notation, we can denote the set of natural numbers as

$$\mathcal{N} = \{x| x \text{ is an integer and } x > 0\},$$

or more elegantly as

$$\mathcal{N} = \{x| x \in \mathcal{Z} \text{ and } x > 0\}.$$ 

Note that some texts use a colon instead of the vertical bar, as in

$$N = \{x : x \in \mathcal{Z} \text{ and } x > 0\}.$$ 

The special set $\{\}$ is called the empty set or sometimes the null or void set. It is denoted by the special symbol $\{\} = \emptyset$. 


Set Relations
Subset. \( A \subseteq B \) is true if for all \( x \in A \), \( x \in S \). \( A \) is a subset of \( B \) if all members of \( A \) are in \( B \).

Equality. \( A = B \) if for all \( x \in A \), \( x \in B \) and for all \( x \in B \), \( x \in A \). Set \( A \) is equal to set \( B \) if all members of \( A \) are in \( B \), and all members of \( B \) are in \( A \). We can also define equality as \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \).

Inequality. \( A \neq B \) if the sets do not have the same members.

Proper Subset. \( A \subset B \) if for all \( x \in A \), \( x \in B \) and \( A \neq B \). Some definitions of proper subset require that \( A \neq \emptyset \).

Set Building Operations
Set Product. \( A \times B = \{ (x, y) | x \in A \text{ and } y \in B \} \). Also called cross product or cartesian product. Elements of the set product are called ordered pairs. Note that coordinates on the Cartesian plane use ordered pairs, with all coordinates \((x, y) \in \mathbb{R} \times \mathbb{R}\).

\[
\{1, 2, 3\} \times \{Y, Z\} = \{(1, Y), (1, Z), (2, Y), (2, Z), (3, Y), (3, Z)\}.
\]

Power Set. Given set \( S \), the power set \( P(S) \) is defined as the set of all subsets of \( S \).

\[
P(\{\}) = \{\}\n\]
\[
P(\{1\}) = \{\}, \{1\}\n\]
\[
P(\{1, 2\}) = \{\}, \{1\}, \{2\}, \{1, 2\}\n\]

Set Size
Given set \( S \), \( |S| \) is the number of elements in \( S \).

\[
|\{\}| = 0
\]
\[
|\{1\}| = 1
\]
\[
|\{1, 2\}| = 2
\]

The notation \( N(A) \), where \( A \) is a set, is sometimes used to denote the size of \( A \).

Set Operations
Given \( U \) an universal set, and sets \( A \subseteq U \) and \( B \subseteq U \), we can define various operators to manipulate sets. The universal set is the universe of possible set elements for the problem domain.

Binary operations have two operands. Unary operators have one operand.

Union. \( A \cup B = \{x | x \in A \text{ or } x \in B\} \). An element is in the union if it is in either operand or both operands.

Intersection. \( A \cap B = \{x | x \in A \text{ and } x \in B\} \). An element is in the union if it is in both operands.

Set Difference. \( A - B = \{x | x \in A \text{ and } x \notin B\} \). An element is in the set difference if it is in the left operand but not the right operand. In effect, set difference “subtracts” elements of the right operand from the left operand.

Symmetric Difference. \( A \triangle B = \{x | x \in A \text{ and } x \notin B \text{ or } x \notin A \text{ and } x \in B\} \). An element is in the symmetric difference if it is in either operand but not in both operands. Note that we can define this as all elements in both sets less the elements in the intersection, \((A \cup B) - (A \cap B)\).
Complement. Various notations are used. The bar notation seems to be the most common. \( \overline{A} = A^c = \neg A = \neg \{x \mid x \in U \text{ and } x \notin A\} \). The complement is the set of elements in the universe that are not in \( A \), \( \overline{A} = U - A \).

Examples. Let \( U = \{0, 1, 2, 3, 4, 5, 6, 7\} \) be the universal set and let \( A = \{1, 3, 5, 7\} \), \( B = \{2, 3, 6, 7\} \), and \( C = \{4, 5, 6, 7\} \).

\[
\begin{align*}
A \cup B &= \{1, 2, 3, 5, 6, 7\} \\
A \cap B &= \{3, 7\} \\
A - B &= \{1, 5\} \\
B - A &= \{2, 6\} \\
A \triangle B &= \{1, 2, 5, 6\} \\
\overline{A} &= \{0, 2, 4, 6\}
\end{align*}
\]

We can use parenthesis to determine the precedence of operations.

\[
\begin{align*}
A \cup (B \cap C) &= \{1, 3, 5, 7\} \cup (\{2, 3, 6, 7\} \cap \{4, 5, 6, 7\}) \\
&= \{1, 3, 5, 7\} \cup \{6, 7\} \\
&= \{1, 3, 5, 6, 7\}
\end{align*}
\]

\[
\begin{align*}
(A \cup B) \cap C &= (\{1, 3, 5, 7\} \cup \{2, 3, 6, 7\}) \cap \{4, 5, 6, 7\} \\
&= \{1, 2, 3, 5, 6, 7\} \cap 4, 5, 6, 7 \\
&= \{5, 6, 7\}
\end{align*}
\]

Other Definitions

Sets \( A \) and \( B \) are disjoint if \( A \cap B = \emptyset \).

The set \( A = \{A_1, A_2, \ldots, A_n\} \) is a partition of set \( S \), if each \( A_i \neq \emptyset \),

\[
\bigcup_{A_i \in A} A_i = S,
\]

and \( A_i \cap A_j = \emptyset \) for all \( A_i, A_j \in A \). A partition breaks a set \( S \) into mutually disjoint, nonempty subsets, where the union of the subsets is the set \( S \).

Counting.

Size of union. If sets \( A \) and \( B \) are disjoint, then \( |A \cup B| = |A| + |B| \).

Size of set product. For sets \( A \) and \( B \), \( |A \times B| = |A| \cdot |B| \).

Size of powerset. For set \( A \), \( |P(A)| = 2^{|A|} \).