Sets

Section 2.1
Sets

- A set is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the elements, or members of the set. A set is said to contain its elements.
- The notation \( a \in A \) denotes that \( a \) is an element of the set \( A \).
- If \( a \) is not a member of \( A \), write \( a \notin A \)
Describing a Set: Roster Method (listing the members)

- \( S = \{a,b,c,d\} \)
- Order not important
  \[ S = \{a,b,c,d\} = \{b,c,a,d\} \]
- Each distinct object is either a member or not; listing more than once does not change the set.
  \[ S = \{a,b,c,d\} = \{a,b,c,b,c,d\} \]
- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.
  \[ S = \{a,b,c,d, \ldots, z\} \]
Some Important Sets

\[ N = \text{natural numbers} = \{0,1,2,3\ldots\} \]
\[ Z = \text{integers} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\} \]
\[ Z^+ = \text{positive integers} = \{1,2,3,\ldots\} \]
\[ R = \text{set of real numbers} \]
\[ R^+ = \text{set of positive real numbers} \]
\[ C = \text{set of complex numbers}. \]
\[ Q = \text{set of rational numbers} \]
Set-Builder Notation

• Specify the property or properties that all members must satisfy:
  \( S = \{ x \mid x \text{ is a positive integer less than } 100 \} \)
  \( O = \{ x \mid x \text{ is an odd positive integer less than } 10 \} \)
  \( O = \{ x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10 \} \)

• A predicate may be used:
  \( S = \{ x \mid P(x) \} \)

• Example: \( S = \{ x \mid \text{Prime}(x) \} \)

• Positive rational numbers:
  \( \mathbb{Q}^+ = \{ x \in \mathbb{R} \mid x = p/q, \text{ for some positive integers } p, q \} \)
Interval Notation

\[ [a,b] = \{x \mid a \leq x \leq b \} \]
\[ [a,b) = \{x \mid a \leq x < b \} \]
\[ (a,b] = \{x \mid a < x \leq b \} \]
\[ (a,b) = \{x \mid a < x < b \} \]

*closed interval* \([a,b]\)

*open interval* \((a,b)\)
Universal Set and Empty Set

- The *universal set* $U$ is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized $\emptyset$, but {} also used.
Some things to remember

- Sets can be elements of sets.
  \[
  \{\{1,2,3\}, a, \{b,c\}\}
  \{N,Z,Q,R\}
  \]
- The empty set is different from a set containing the empty set.
  \[\emptyset \neq \{\emptyset\}\]
Set Equality

**Definition:** Two sets are *equal* if and only if they have the same elements.

- Therefore if $A$ and $B$ are sets, then $A$ and $B$ are equal if and only if $\forall x (x \in A \iff x \in B)$.
- We write $A = B$ if $A$ and $B$ are equal sets.

\[
\{1, 3, 5\} = \{3, 5, 1\}
\]
\[
\{1, 5, 5, 5, 3, 3, 1\} = \{1, 3, 5\}
\]
Subsets

Definition: The set $A$ is a *subset* of $B$, if and only if every element of $A$ is also an element of $B$.

- The notation $A \subseteq B$ is used to indicate that $A$ is a subset of the set $B$.
- $A \subseteq B$ holds if and only if $\forall x (x \in A \rightarrow x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set $S$.
  2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set $S$. 
Another look at Equality of Sets using Subsets

- Recall that two sets $A$ and $B$ are equal, denoted by $A = B$, iff
  \[ \forall x (x \in A \iff x \in B) \]

- Using logical equivalences we have that $A = B$ iff
  \[ \forall x [(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)] \]

- This is equivalent to
  \[ A \subseteq B \quad \text{and} \quad B \subseteq A \]
**Set Cardinality**

**Definition:** If there are exactly $n$ distinct elements in $S$ where $n$ is a nonnegative integer, we say that $S$ is *finite*. Otherwise it is *infinite*.

**Definition:** The *cardinality* of a finite set $A$, denoted by $|A|$, is the number of (distinct) elements of $A$.

**Examples:**

1. $|\emptyset| = 0$
2. Let $S$ be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.
Power Sets

Definition: The set of all subsets of a set $A$, denoted $P(A)$, is called the power set of $A$.

Example: If $A = \{a,b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

- If a set has $n$ elements, then the cardinality of the power set is $2^n$. 

Tuples

• The ordered n-tuple \((a_1,a_2,\ldots,a_n)\) is the ordered collection that has \(a_1\) as its first element and \(a_2\) as its second element and so on until \(a_n\) as its last element.

• Two n-tuples are equal if and only if their corresponding elements are equal.

• 2-tuples are called ordered pairs.

• The ordered pairs \((a,b)\) and \((c,d)\) are equal if and only if \(a = c\) and \(b = d\).
**Cartesian Product**

**Definition:** The *Cartesian Product* of two sets $A$ and $B$, denoted by $A \times B$ is the set of ordered pairs $(a,b)$ where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \land b \in B\}$$

**Example:**

$A = \{a, b\}$  $B = \{1, 2, 3\}$

$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
Truth Sets of Quantifiers

- Given a predicate $P$ and a domain $D$, we define the truth set of $P$ to be the set of elements in $D$ for which $P(x)$ is true. The truth set of $P(x)$ is denoted by

$$\{x \in D | P(x)\}$$

- **Example:** The truth set of $P(x)$ where the domain is the integers and $P(x)$ is “$|x| = 1$” is the set $\{-1,1\}$
Set Operations

Section 2.2
Union

- **Definition:** Let $A$ and $B$ be sets. The *union* of the sets $A$ and $B$, denoted by $A \cup B$, is the set:
  \[ \{ x | x \in A \lor x \in B \} \]

- **Example:** What is \( \{1,2,3\} \cup \{3,4,5\} \)?

  **Solution:** \( \{1,2,3,4,5\} \)
Intersection

- **Definition**: The *intersection* of sets $A$ and $B$, denoted by $A \cap B$, is
  \[
  \{x \mid x \in A \land x \in B\}
  \]

- Note if the intersection is empty, then $A$ and $B$ are said to be *disjoint*.

- **Example**: What is? \{1,2,3\} $\cap$ \{3,4,5\} ?
  
  **Solution**: \{3\}

- **Example**: What is? \{1,2,3\} $\cap$ \{4,5,6\} ?
  
  **Solution**: $\emptyset$
Complement

**Definition:** If \( A \) is a set, then the complement of the \( A \) (with respect to \( U \)), denoted by \( \bar{A} \) is the set \( U - A \)

\[
\bar{A} = \{x \in U \mid x \notin A\}
\]

(The complement of \( A \) is sometimes denoted by \( A^c \).)

**Example:** If \( U \) is the positive integers less than 100, what is the complement of \( \{x \mid x > 70\} \)

**Solution:** \( \{x \mid x \leq 70\} \)

Venn Diagram for Complement
Difference

**Definition:** Let $A$ and $B$ be sets. The *difference* of $A$ and $B$, denoted by $A - B$, is the set containing the elements of $A$ that are not in $B$. The difference of $A$ and $B$ is also called the complement of $B$ with respect to $A$.

$$A - B = \{x \mid x \in A \land x \notin B\} = A \cap B^c$$

[Venn Diagram for $A - B$]
The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

- Cardinality – number of *unique* elements – the Venn diagram makes it easy to see why we need that last term.
Set Identities

- Identity laws
  \[ A \cup \emptyset = A \quad A \cap U = A \]
- Domination laws
  \[ A \cup U = U \quad A \cap \emptyset = \emptyset \]
- Idempotent laws
  \[ A \cup A = A \quad A \cap A = A \]
- Complementation law
  \[ (\overline{A}) = A \]

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Set Identities

- Commutative laws

\[ A \cup B = B \cup A \quad A \cap B = B \cap A \]

- Associative laws

\[ A \cup (B \cup C) = (A \cup B) \cup C \]
\[ A \cap (B \cap C) = (A \cap B) \cap C \]

- Distributive laws

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

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Set Identities

- **De Morgan’s laws**
  \[ A \cup B = \overline{A} \cap \overline{B} \quad A \cap B = \overline{A} \cup \overline{B} \]

- **Absorption laws**
  \[ A \cup (A \cap B) = A \quad A \cap (A \cup B) = A \]

- **Complement laws**
  \[ A \cup \overline{A} = U \quad A \cap \overline{A} = \emptyset \]
Proving Set Identities

- The most common way to prove set identities:
  - Prove that each set (side of the identity) is a subset of the other.
Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$
Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$ by assumption
$x \notin A \cap B$ defn. of complement
$\neg((x \in A) \land (x \in B))$ defn. of intersection
$\neg(x \in A) \lor \neg(x \in B)$ 1st De Morgan Law for Prop Logic
$x \notin A \lor x \notin B$ defn. of negation
$x \in \overline{A} \lor x \in \overline{B}$ defn. of complement
$x \in \overline{A} \cup \overline{B}$ defn. of union

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Proof of Second De Morgan Law

These steps show that:

\[ \overline{A \cup B} \subseteq \overline{A \cap B} \]

- \( x \in \overline{A \cup B} \) by assumption
- \((x \in \overline{A}) \lor (x \in \overline{B})\) defn. of union
- \((x \notin A) \lor (x \notin B)\) defn. of complement
- \(\neg(x \in A) \lor \neg(x \in B)\) defn. of negation
- \(\neg((x \in A) \land (x \in B))\) by 1st De Morgan Law for Prop Logic
- \(\neg(x \in A \cap B)\) defn. of intersection
- \(x \in \overline{A \cap B}\) defn. of complement