CSC/MTH447 Fall 2012
Midterm #2
Version 1.0

Take-Home Exam

Due Wednesday 11/28/12 in Class

Name: ______________________________

- Below you will find five (5) questions each worth 20 points.
- Please answer all of them.
- Please show all your work. If you get stuck on a problem at least give me an outline on the approach you were planning on taking. Partial credit is given.
- Please answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

1. Do the following:

   (a) (10 points) Prove that $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

   (b) (10 points) For which nonnegative integers n is $n^2 \leq n!$? Prove your answer.

2. (a) (10 points) Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

Use **structural induction** to show that '5 divides $(a + b)$' when $(a, b) \in S$.

(b) (10 points) Recursively define set expressions with variables representing sets and operators drawn from $\{\cup, \cap, -\}$. Then use structural induction to show that $O \leq V$ for all possible set expressions with O the number of operators in an expression and V the number of variables in an expression.

3. There are four possibilities for each base in DNA: A, C, G, and T. How many 5-element DNA sequences

   (a) (5 points) end with A?

   (b) (5 points) start with T and end with G?

   (c) (5 points) contain only A and T?

   (d) (5 points) do not contain C?

4. (a) (5 points) Suppose that the function f from A to B is a one-to-one correspondence. Let R be the relation that equals the graph of f. That is, $R = \{(a, f(a)) | a \in A\}$. What is the inverse relation R^{-1}?

   (b) (5 points) Show that the relation $R = \emptyset$ on a nonempty set S is symmetric and transitive, but not reflexive.

   (c) Suppose that R and S are reflexive relations on a set A. Prove or disprove each of the following statements:

      (a) (5 points) $R \cup S$ is reflexive.

      (b) (5 points) $S \circ R$ is reflexive.

5. (a) (10 points) Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

(b) Suppose that R_1 and R_2 are equivalence relations on the set S. Determine whether the following sets are equivalence relations.

(a) (5 points) $R_1 \cup R_2$

(b) (5 points) $R_1 \cap R_2$