CSC 212
Data Structures and Abstractions
Spring 2016

Lecture 12: Binary Search Trees III
Previously ...

Binary Search Trees

- insertion
- deletion
- traversals
BST’s delete

Case 1: node is a leaf
trivial, delete node and set parent’s pointer to NULL

Case 2: node has 1 child
trivial, set parent’s pointer to the only child and delete node

Case 3: node has 2 children
find successor and copy successor’s data to node
delete successor
Analysis
Tree Shape

Depends on order of insertion
Implications

Cost of basic operations?
search, insert, remove
Implications

Cost of basic operations?
search, insert, remove

worst-case
Implications

Cost of basic operations?
  search, insert, remove

worst-case $O(h) = O(n)$
Implications

Cost of basic operations?
search, insert, remove

worst-case \( O(h) = O(n) \)
best-case
Implications

Cost of basic operations?
search, insert, remove

**worst-case** \( O(h) = O(n) \)

**best-case** \( O(h) = O(\log n) \)
Implications

Cost of basic operations?
  search, insert, remove

worst-case \( O(h) = O(n) \)

best-case \( O(h) = O(\log n) \)

average-case
Implications

Cost of basic operations?
    search, insert, remove

<table>
<thead>
<tr>
<th>Case</th>
<th>Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>worst-case</td>
<td>$O(h) = O(n)$</td>
</tr>
<tr>
<td>best-case</td>
<td>$O(h) = O(\log n)$</td>
</tr>
<tr>
<td>average-case</td>
<td>$O(h) = ?$</td>
</tr>
</tbody>
</table>
Correspondence with QuickSort

If $n$ distinct keys are inserted into a BST in random order
Correspondence with QuickSort

If $n$ distinct keys are inserted into a BST in random order

expected number of compares for a search/insert is

$\sim 2 \ln n \sim = 1.39 \log n$
Correspondence with QuickSort

If \( n \) distinct keys are inserted into a BST in random order

expected number of compares for a search/insert is

\(~2 \ln n \approx 1.39 \log n\)

**proof**: 1-1 correspondence with quick-sort partitioning
Correspondence with QuickSort

If $n$ distinct keys are inserted into a BST in random order

expected number of compares for a search/insert is

$\sim 2 \ln n \sim = 1.39 \log n$

**proof**: 1-1 correspondence with quick-sort partitioning

**Average Case**: $O(\log n)$
N = 255
max = 16
avg = 9.1
opt = 7.0

The Height of a Random Binary Search Tree

BRUCE REED

McGill University, Montreal Quebec, Canada and CNRS, Paris, France

Abstract. Let $H_n$ be the height of a random binary search tree on $n$ nodes. We show that there exist constants $\alpha = 4.311 \cdots$ and $\beta = 1.953 \cdots$ such that $E(H_n) = \alpha \ln n - \beta \ln \ln n + O(1)$, We also show that $\text{Var}(H_n) = O(1)$.

Categories and Subject Descriptors: E.1 [Data Structures]: trees; G.2 [Discrete Mathematics]; G.3 [Probability and Statistics]

General Terms: Algorithms, Theory

Additional Key Words and Phrases: Binary search tree, height, probabilistic analysis, random tree, asymptotics, second moment method
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Expected height: $\sim 4.311 \ln n \sim = 2.99 \log n$
Other Operations

Min() — find smallest value

Max() — find largest value

Floor(k) — find largest value \( \leq \) than \( k \)

Ceiling(k) — find smallest value \( \geq \) than \( k \)
Other Operations

Min() — find smallest value
Max() — find largest value
Floor(k) — find largest value <= than k
Ceiling(k) — find smallest value >= than k
Other Operations

Min() — find smallest value

Max() — find largest value

Floor(k) — find largest value $\leq$ than k

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Other Operations

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## Computational Cost

<table>
<thead>
<tr>
<th></th>
<th>sequential search (unordered sequence)</th>
<th>binary search (ordered sequence)</th>
<th>Binary Search Tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>search</td>
<td>O(n)</td>
<td>O(log n)</td>
<td>O(h)</td>
</tr>
<tr>
<td>insert</td>
<td>O(n)</td>
<td>O(n)</td>
<td>O(h)</td>
</tr>
<tr>
<td>delete</td>
<td>O(n)</td>
<td>O(n)</td>
<td>O(h)</td>
</tr>
<tr>
<td>min/max</td>
<td>O(n)</td>
<td>O(1)</td>
<td>O(h)</td>
</tr>
<tr>
<td>floor/ceiling</td>
<td>O(n)</td>
<td>O(log n)</td>
<td>O(h)</td>
</tr>
<tr>
<td>rank</td>
<td>O(n)</td>
<td>O(log n)</td>
<td>O(h) **</td>
</tr>
</tbody>
</table>

** requires the use of ‘size’ at every node
Can we sort using BSTs?

Given \( n \) numbers …

what is the cost of a bad case?

what is the cost of a best case?
Can we sort using BSTs?

Given $n$ numbers …

what is the cost of a bad case? $O(n^2)$

what is the cost of a best case?
Can we sort using BSTs?

Given $n$ numbers …

what is the cost of a bad case? $O(n^2)$

what is the cost of a best case? $O(n \log n)$