CSC 212: Data Structures and Abstractions
07: Lower Bounds on Sorting, Stability

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Sorting based on Comparisons

- Basic operation: compare two items

- Consider sorting three items (x, y, z)
  - how many comparisons are needed (at least)?

- Consider sorting n items
  - is there a lower bound?

Decision Tree (sorting x, y, z)

- What is the worst-case number of comparisons?
  - height of the decision tree (length of longest path from root to a leave)

- Consider sorting n distinct items
  - what is the height?
  - ... use the number of leaves

  number of leaves at least \( n! \)
  number of leaves at most \( 2^h \)

# of all permutations
perfect binary tree
What is the height?

\[
2^h \geq \# \text{ leaves} \geq n!
\]
\[
2^h \geq n!
\]
\[
\log 2^h \geq \log n! \quad \text{... by Stirling’s formula}
\]
\[
h \geq n \log n
\]

Cost of Sorting

\[
\Omega(n \log n)
\]

What is the lower bound for the cost of sorting algorithms based on comparisons?

What is the cost of sorting algorithms considered optimal?

\[
\Theta(n \log n)
\]

No sorting algorithm based on key comparisons can possibly be faster than \(\Omega(n\log n)\) in the worst case.

Problem

Sort a flight departures table:

- by time then by location

Stability

Stability when sorting on a second key

http://algs4.cs.princeton.edu/25applications/
Stability

• A sorting algorithm is **stable** if it preserves the order of equal elements

![Stable Sorting Example](image)

Stability

• Is selection sort stable? 🙅
  • long distance swaps
  • try: 5 1 2 4 4 3 2 1

• Is insertion sort stable? 👍
  • equal items never pass each other (depends on correct implementation)

In-place?

• A sorting algorithm is **in-place** if it uses $O(\log n)$ extra memory

• Selection and Insertion sorts are **in-place**

Memory Analysis
## Summary

<table>
<thead>
<tr>
<th></th>
<th>Best-Case</th>
<th>Average-Case</th>
<th>Worst-Case</th>
<th>Stable?</th>
<th>In-place?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selection Sort</td>
<td>$\theta(n^2)$</td>
<td>$\theta(n^2)$</td>
<td>$\theta(n^2)$</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Insertion Sort</td>
<td>$\theta(n)$</td>
<td>$\theta(n^2)$</td>
<td>$\theta(n^2)$</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>