Sorting based on Comparisons

- Basic operation: compare two items

- Consider sorting three items \((x, y, z)\)
  - how many comparisons are needed (at least)?

- Consider sorting \(n\) items
  - is there a lower bound?

Decision Tree (sorting \(x, y, z\))

- What is the worst-case number of comparisons?
  - height of the decision tree (length of longest path from root to a leave)

- Consider sorting \(n\) distinct items
  - what is the height? 😐
  - … can use the number of leaves

  - number of leaves at least \(n!\)
    - # of all permutations
  - number of leaves at most \(2^h\)
    - perfect binary tree

- Consider sorting \(n\) items
  - is there a lower bound?
What is the height?

\[
2^h \geq \# \text{ leaves} \geq n!
\]

\[
2^h \geq n!
\]

\[
\log 2^h \geq \log n!
\]

\[
h \geq n \log n \quad \text{... by Stirling's formula}
\]

Cost of Sorting

\cdot What is a lower bound for the cost of sorting algorithms based on comparisons?

\[ \Omega(n \log n) \]

\cdot What is the cost of sorting algorithms considered optimal?

\[ \Theta(n \log n) \]

No sorting algorithm based on key comparisons can possibly be faster than \( \Omega(n \log n) \) in the worst case.

Need a Break?

https://www.youtube.com/watch?v=XZmGGAHqa0

Stability
Problem

- Sort a flight departures table
  - by time then by location

Stability

- A sorting algorithm is **stable** if it preserves the order of equal elements

Stability

- Is selection sort stable? **✗**
  - long distance swaps
  - try: 5 1 2 4 4 3 2 1

- Is insertion sort stable? **✓**
  - equal items never pass each other (depends on correct implementation)
In-place?

- A sorting algorithm is **in-place** if it uses $O(\log n)$ extra memory

- Are selection and insertion sorts **in-place**?

### Summary

<table>
<thead>
<tr>
<th></th>
<th>Best-Case</th>
<th>Average-Case</th>
<th>Worst-Case</th>
<th>Stable?</th>
<th>In-place?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Selection Sort</strong></td>
<td>$\Theta(n^2)$</td>
<td>$\Theta(n^2)$</td>
<td>$\Theta(n^2)$</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td><strong>Insertion Sort</strong></td>
<td>$\Theta(n)$</td>
<td>$\Theta(n^2)$</td>
<td>$\Theta(n^2)$</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>