Probability, as a subject, provides a means of quantifying uncertainty.

4.2 Probability of an Event

Experiment

• Is a process that results in

• The outcome that occurs

Example

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flip a coin</td>
<td></td>
</tr>
<tr>
<td>Roll a die</td>
<td></td>
</tr>
<tr>
<td>Dow Jones</td>
<td></td>
</tr>
</tbody>
</table>
Actual outcome can not be determined (predicted) in advance;

but

can assign probabilities to outcomes

Sample space

List of experiment.

Denote the sample space by $S$.

i.e. Coin: $S =$
Elementary outcome

Each outcome.

Denote the outcomes by $e_1, e_2, \ldots$

i.e. Coin:

Event

A collection

i.e. Dow Jones doesn't increase.

$\rightarrow \{ \}$

Denote events with capital letters: $A, E_i, \ldots$
Meaning of Probability

Probability is

• a generalization of the concept of

• a numerical measure of how
Properties of Probabilities - A is an event.

- $P(A) = 1 \rightarrow$ certain event
- $P(A) = 0 \rightarrow$ impossible event
- $P(A) = \quad$
- $P(S) = \quad$
4.4 Events relations and two laws of probabilities

Venn Diagrams

Useful for illustrating probability concepts.

- Sample space is depicted as

- Events are drawn as

Venn diagram for event A

![Venn diagram for event A](image-url)
Union

The union of any two events A and B, denoted $A \cup B$ (A or B), occurs if

Example - Tossing a die.

$C = \{\} \text{ (Even numbers)}$

$D = \{\} \text{ (Less than or = 3)}$

$C \cup D =$
Intersection

The intersection of any two events $A$ and $B$, denoted $A \cap B$ (AB) occurs only

Example

$C \cap D =$
Mutually Exclusive (Incompatible events)

Two events A and B are mutually exclusive if

\[ A \cap B = \]

Example

\[ C = \{ \} \quad \text{(Even numbers)} \]
\[ F = \{ \} \quad \text{(Odd numbers)} \]

\[ C \cap F = \]
Complement

The **complement** of any event $A$, denoted $\bar{A}$, is the set of all simple events.

Example - $F = \{1, 3, 5\}$ (die) $\bar{F} = \{\}$

For any event $A$,

1. $A \cap \bar{A} =$
2. $A \cup \bar{A} =$ (Exhaustive).
Probability Laws:

- **Addition Law**
  (Mutually Exclusive Events) :

  When A and B are mutually exclusive

  \[ P(A \cup B) = \]

  **Ex.** Die: A={1} and B={2,4}

  \[ P(A \cup B) = \]
  
  \[ = 3/6 = 0.5 \]

Note: The addition rule can be extended to more than 2 events.
• Addition Law (General)

Events A and B do not have to be mutually exclusive events.

\[ P(A \cup B) = \]

Ex. A: even number tossed
B: number < 3 tossed

\[ P(A \cup B) = \]

\[ = \]

\[ = \frac{4}{6} \]
• Complement Rule

\(\bar{A}\) : Event that A does not occur and we know that

\[A \cup \bar{A} = S \rightarrow P(A \cup \bar{A}) = \]

hence \[\text{ _______ } = \]

Also:

\[
\begin{align*}
P(A) &= 1 - P(\bar{A}) \\
P(\bar{A}) &= 1 - P(A)
\end{align*}
\]

Ex. Assuming \(P(\bar{A}) = 2/6\), find the \(P(A)\).

Using the complement rule

\[P(A) = \text{ _______ } = 4/6\]
Ex. As reported in Employment and Earnings, the age dist’n of employed persons 16 years old and over is:

<table>
<thead>
<tr>
<th>Age</th>
<th>Frequency (000's)</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>16-19</td>
<td>6,500</td>
<td>A</td>
</tr>
<tr>
<td>20-24</td>
<td>12,138</td>
<td>B</td>
</tr>
<tr>
<td>25-34</td>
<td>32,077</td>
<td>C</td>
</tr>
<tr>
<td>35-44</td>
<td>35,051</td>
<td>D</td>
</tr>
<tr>
<td>45-54</td>
<td>25,514</td>
<td>E</td>
</tr>
<tr>
<td>55-64</td>
<td>11,739</td>
<td>F</td>
</tr>
<tr>
<td>65 &amp; over</td>
<td>3,690</td>
<td>G</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>126,709</strong></td>
<td></td>
</tr>
</tbody>
</table>

An employed person is selected at random. Let the following events be defined:

- \( W = \) the person is between 20 and 64
- \( Y = \) the person is under 65
- \( Z = \) the person is 55 or over.

Describe each of the following events in words and determine their probabilities.
a. (not Y)

Is the event the person selected is

\[ P \text{ (not Y)} = P (\quad) \]

\[ = P (\quad) \]

\[ = P (\quad) \]

\[ = 0.0291 \]

\[ P \text{ (Y)} = ? \]

Could use the

\[ P \text{ (Y)} = \]

\[ = 0.9709 \]
b. (not W)

Is the event the person selected is

\[ P(\bar{W}) = P(\quad) \]

\[ = P(\quad) \]

Since Mutually Exclusive

\[ = \]

\[ = + \]

\[ = 0.0804 \]
c. W and Z (same as $W \cap Z$ or $WZ$)

Is the event the person selected is

$$P (W \cap Z) = P(WZ)$$

$$= P( )$$

$$= 0.0926$$
d. W or Z

Is the event that the person selected

Are W and Z mutually exclusive?

Since \( W \cap Z = \)
\( \rightarrow W \) and Z are

Thus,

\[ P(W \cup Z) = \]

But, \( P(W) = \)

\[ = \quad = \quad = 0.9196 \]

\[ P(Z) = \]

\[ = 0.1218 \]
and

\[ P(W \cap Z) = \]

Hence,

\[ P(W \cup Z) = \]

\[ = 0.9196 \quad 0.1218 \quad 0.0926 \]

\[ = 0.9488 \]

**Easier** - use complement rule!

\[ P(W \cup Z) = \]

\[ = 1 - \frac{6500}{126,709} = 0.9487 \]
4.5 Conditional Probability & Independence

The **conditional probability** \( P(A \mid B) \) means the probability that \( A \) will occur given that \( B \) has occurred.

**Ex.** Select randomly from standard pack of 52 cards (no replacement). Let

\[ A_1 : \text{1st card is an ace} \]
\[ A_2 : \text{2nd card is an ace} \]

\[
P(A_1) = ?
\]

\[
P(A_2 \mid A_1) = ?
\]
Ex. Throw a die

\[ L = \{1, 2, 3\} \quad \text{and} \quad M = \{1, 2, 3, 4\} \]

What is \( P(L \mid M) = ? \)

Reduced sample space:

\[ M = \{1, 2, 3, 4\} \]

\[ P(L \mid M) = \]

**Alternatively:**

\[ \frac{\text{No. of common outcomes in } L \text{ and } M}{\text{No. outcomes in } M} \]

\[ \rightarrow P(L \mid M) = \]
General Formula:

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0
\]

Ex. \( P(M) = \)

\[
P(L \cap M) =
\]

\[
P(L \mid M) = \frac{3}{4} = .75
\]
Multiplication Rule

To find the probability of a joint event i.e. \( P(A \cap B) \), for any two events \( A \) and \( B \), rearrange the conditional probability rule

\[
P(A | B) = \frac{P(A \cap B)}{P(B)}.
\]

Multiplying both sides by \( P(B) \) gives

\[
P(A \cap B) = P(A | B) \times P(B).
\]

Alternatively:

\[
P(B | A) = \frac{P(A \cap B)}{P(A)}
\]

\[
\rightarrow P(A \cap B) = P(B | A) \times P(A)
\]

Ex. \[
P(L \cap M) =
\]

\[
= \frac{1}{2}
\]
Independent Events

Two events A and B are said to be independent if

\[
P(A \mid B) = \quad \text{or} \quad P(B \mid A) = \quad .
\]

Otherwise the events are dependent.

Example

\[P(L \mid M) = \frac{3}{4} \quad \text{but} \quad P(L) = \frac{3}{6}\]

\[P(L \mid M) \times P(L)\]

so L and M are

Multiplication Rule (Independent Events)

With independent events \(P(A \mid B) = P(A)\) hence, the multiplication rule then reduces to

\[P(A \cap B) = P(A) \times P(B).\]
Ex. According to *Census of Agriculture*, a joint frequency dist’n for the number of farms, by acreage and tenure of operator, is as shown in the following contingency table. Frequencies are in thousands.

<table>
<thead>
<tr>
<th>Acreage: X</th>
<th>Tenure of operator</th>
<th>Full Owner $T_1$</th>
<th>Part Owner $T_2$</th>
<th>Tenant $T_3$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X &lt; 50$</td>
<td>$A_1$</td>
<td>444</td>
<td>58</td>
<td>52</td>
<td>554</td>
</tr>
<tr>
<td>$50 \leq X &lt; 180$</td>
<td>$A_2$</td>
<td>395</td>
<td></td>
<td>59</td>
<td>584</td>
</tr>
<tr>
<td>$180 \leq X &lt; 500$</td>
<td>$A_3$</td>
<td>190</td>
<td></td>
<td>55</td>
<td>428</td>
</tr>
<tr>
<td>$500 \leq X &lt; 1000$</td>
<td>$A_4$</td>
<td>48</td>
<td>111</td>
<td></td>
<td>186</td>
</tr>
<tr>
<td>$1000 &amp; over$</td>
<td>$A_5$</td>
<td>35</td>
<td>114</td>
<td>24</td>
<td>173</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td></td>
<td>1112</td>
<td>596</td>
<td>217</td>
<td>1925</td>
</tr>
</tbody>
</table>
a. Fill in the three empty cells

   Missing frequencies in cells

b. How many cells does the contingency table have?

c. Find the probability $P(T_1)$ and $P(A_1)$

   
   $P(T_1) = 0.578$

   
   $P(A_1) = 0.288$
d. Find the probability $P(T_1 \cap A_1)$

$$P(T_1 \cap A_1) = 0.231$$

e. Obtain $P(A_1|T_1)$ directly from the table

$$P(A_1|T_1) = 0.399$$

f. Obtain $P(A_1|T_1)$ using the conditional probability rule and your answers from part (c) and (d).

$$P(A_1|T_1) = \frac{444}{1,112} = 0.399$$
g. Are $A_1$ and $T_1$ independent? Explain.

\[ P(A_1 \cap T_1) = \frac{0.399}{0.288} = 0.546 \]

Hence, $A_1$ and $T_1$ are not independent.

Exploring dependency further:

\[ P(A_1 | T_2) = \frac{0.012}{0.10} = 0.12 \]

\[ P(A_1 | T_3) = \frac{0.02}{0.06} = 0.33 \]

There is tendency for
h. Obtain $P(T_1 \cap A_1)$ using the multiplication rule and your answers from parts (c) and (e).

$$P(T_1 \cap A_1) =$$

$$= \times = 0.231$$

4.6 & 4.7 Skip these sections