CSC/MTH 447 Discrete Mathematics

Dr. Lutz Hamel
hamel@cs.uri.edu
Tyler Hall 251

Welcome!

What is Discrete Mathematics?

- Discrete mathematics is the part of mathematics devoted to the study of discrete (as opposed to continuous) objects.
- Calculus deals with continuous objects and is not part of discrete mathematics.
- Examples of discrete objects: integers, steps taken by a computer program, distinct paths to travel from point A to point B on a map along a road network, ways to pick a winning set of numbers in a lottery.

Goals of the Course

- **Mathematical Reasoning**: Ability to read, understand, and construct mathematical arguments and proofs.
- **Combinatorial Analysis**: Techniques for counting objects of different kinds.
- **Discrete Structures**: Abstract mathematical structures that represent objects and the relationships between them. Examples are sets, permutations, relations, graphs, trees, and finite state machines.

The Foundations: Logic and Proofs

A Formal Language

- We need a language with precise rules in order to reason about mathematical objects
- Logic is such a language because the rules of logic give precise meaning to mathematical statements
- We can use these rules to distinguish between valid and invalid mathematical arguments.
- We call a language with such precise rules a *formal language* in order to distinguish it from our natural languages (calling logic an unnatural language just doesn't sound right)

Formal Language

- Formal languages are also used heavily in computer science:
    - System specification and verification
    - Knowledge databases and AI
    - Design of computer circuits
    - Automatic Theorem Provers

Formal Language

- One of the simplest form of formal language and logic we can study is Propositional Logic - the logic of declarative sentences.
- The nice thing about Propositional Logic is that it generalizes very nicely to First-Order Logic (FOL is powerful enough to formalize all of mathematics!)

Propositional Logic

Section 1.1

Propositions

- A *proposition* is a declarative sentence that is either true or false.
- Examples of propositions:
  a) The Moon is made of green cheese.
  b) Trenton is the capital of New Jersey.
  c) Toronto is the capital of Canada.
  d) $1 + 0 = 1$
  e) $0 + 0 = 2$
- Examples that are not propositions.
  a) Sit down!
  b) What time is it?
  c) $x + 1 = 2$
  d) $x + y = z$

Propositional Logic

- Elements of Propositional Logic
  - Propositional Variables: *p*, *q*, *r*, *s*, …
    - The value of a propositional variable is either T (true) or F (false)
  - The proposition that is always true is denoted by **T**.
  - The proposition that is always false is denoted by **F**.

Propositional Logic

- Elements of Propositional Logic
  - Compound Propositions; constructed from logical connectives and other propositions
    - Negation ¬
    - Conjunction ∧
    - Disjunction ∨
    - Implication →
    - Biconditional ↔
  - Example: if p and q are propositions, then so is p ∧ q
  - We can discover the value of a compound proposition by constructing *truth tables.*

Compound Propositions: Negation

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

- **Example**: If p denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Conjunction

- The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Disjunction

- The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

The Connective Or in English

- In English "or" has two distinct meanings.
  - "Inclusive Or" - In the sentence "Students who have taken CS202 or Math120 may take this class," we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \lor q$ to be true, either one or both of p and q must be true.
  - "Exclusive Or" - When reading the sentence "Soup or salad comes with this entrée," we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes "If I am at home then it is raining."
- In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- One way to view the logical conditional is to think of an obligation or contract.
  - "If you get 100% on the final, then you will get an A."
- Let p denote "you get 100% on the final" and let q denote "you get an A"
- Note that the contract is broken only in the second line of the truth table.
- Also note that the contract does not say anything about if you don't get 100% on the final, you might still get an A but you might not.

Different Ways of Expressing $p \rightarrow q$

- **if** p**, then** q
- **if** p**,** q
- q **unless** $\neg p$
- q **if** p
- q **whenever** p
- q **follows from** p
- p **implies** q
- p **only if** q
- q **when** p
-
- p **is sufficient for** q
- q **is necessary for** p

- **a necessary condition for** p **is** q
- **a sufficient condition for** q **is** p

Different Ways of Expressing $p \to q$

p	$\neg p$	q	$p \to q$	r
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

- Let proposition r be:
  - q is necessary for p
  - a necessary condition for p is q
  - p only if q
  - q unless ¬p (≅ q ∨ ¬p)

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example: Find the converse, inverse, and contrapositive of "It raining is a sufficient condition for my not going to town."

Solution:

converse: If I do not go to town, then it is raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

Biconditional

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q ." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$	$p \rightarrow q \land q \rightarrow p$
T	T	T	
T	F	F	
F	T	F	
F	F	T	

- Example: If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways "*p* if and only if *q*" is expressed in English:

  - ***p* is necessary and sufficient for *q***
  - **if *p* then *q* , and conversely**
  - ***p* iff *q***

Truth Tables For Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
    - This includes the atomic propositions

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are **e***quivalent* if they always have the same truth value.
- **Example**: Show using a truth table that the biconditional is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	F	T	T