Propositional Equivalences

Section 1.3

Tautologies, Contradictions, and Contingencies

- A *tautology* is a proposition which is always true.
  - Example: $p \lor \neg p$
- A *contradiction* is a proposition which is always false.
  - Example: $p \land \neg p$
- A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

Definition:

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- Alternatively, two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.

Logically Equivalent

Example:

- This truth table shows $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan

1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

- Identity Laws: $p \wedge T \equiv p$, $p \vee F \equiv p$
- Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv F$
- Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$
- Double Negation Law: $\neg(\neg p) \equiv p$
- Negation Laws: $p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$

Key Logical Equivalences (*cont*)

- Commutative Laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$
- Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
  $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive Laws: $(p \vee (q \wedge r) \equiv (p \vee q)) \wedge (p \vee r)$
  $(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$
- Absorption Laws: $p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.
- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$
\begin{aligned}
A &\equiv A_1 \\
&\vdots \\
A_n &\equiv B
\end{aligned}
$$

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$\begin{array}{lll}
\neg(p \vee (\neg p \wedge q)) & \equiv \neg p \wedge \neg(\neg p \wedge q) & \text{by the second De Morgan law} \\
& \equiv \neg p \wedge [\neg(\neg p) \vee \neg q] & \text{by the first De Morgan law} \\
& \equiv \neg p \wedge (p \vee \neg q) & \text{by the double negation law} \\
& \equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \text{by the second distributive law} \\
& \equiv F \vee (\neg p \wedge \neg q) & \text{because } \neg p \wedge p \equiv F \\
& \equiv (\neg p \wedge \neg q) \vee F & \text{by the commutative law} \\
& & \text{for disjunction} \\
& \equiv (\neg p \wedge \neg q) & \text{by the identity law for } \mathbf{F}
\end{array}$$

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Propositional Satisfiability

- A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that makes it true. When no such assignments exist, the compound proposition is *unsatisfiable*.
- A compound proposition is unsatisfiable if and only if its negation is a tautology.

Questions on Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

Solution: Satisfiable. Assign **T** to *p, q,* and *r*.

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Satisfiable. Assign **T** to *p* and ***F*** to *q*.

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.

Solving Satisfiability Problems

- Solving large satisfiability problems can lead to very complex computations…
- A truth table can always be used to determine the satisfiability of a compound proposition. But this is too complex even for modern computers for large problems.
- There has been much work on developing efficient methods for solving satisfiability problems as many practical problems can be translated into satisfiability problems.
- The area of theory of computation studies this in much detail.