Cardinality of Sets

Section 2.5

Cardinality

(another) Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |B|,$$

if and only if there is a bijection from A to B.

- If there is an injection from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.

Cardinality

- **Definition**: A set that is either finite or has the same cardinality as the set of positive integers ($\mathbf{Z}^+$) is called *countable*. A set that is not countable is *uncountable*.

- The set of all finite strings over the alphabet of lowercase letters is countable.
- The set of real numbers **R** is an uncountable set.

Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a bijection f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

Showing that a Set is Countable

Example 1: Show that the set of positive even integers is countable set.

Proof: Let E be the set of even integers and $f(x) = 2x$ be a function from **N** to E.

1	2	3	4	5	6	.....
↕	↕	↕	↕	↕	↕	
2	4	6	8	10	12	......

Then f is a bijection from **N** to E since f is both one-to-one and onto. To show that it is injective, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is surjective, suppose that t is some even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$. ◀

Showing that a Set is Countable

Example 2: Show that the set of all integers **Z** is countable.

Proof: We can list the integers in a sequence:

$0, 1, -1, 2, -2, 3, -3, \ldots\ldots\ldots\ldots$

Let f be a function from **N** to **Z** defined as

- When n is even: $f(n) = n/2$
- When n is odd: $f(\mathrm{n}) = -(n-1)/2$

that generates this list. We now show that this function

is a bijection. First we show that it is injective by case analysis on the parity of **N**.

Showing that a Set is Countable

- Let m and n be two even natural numbers, then $f(m) = m/2$ and $f(n) = n/2$, it follows that $f(m)=f(n)$ implies $m=n$.
- Let m and n be two odd natural numbers, then $f(m) = -(m-1)/2$ and $f(n) = -(n-1)/2$, it follows that $f(m)=f(n)$ implies $m=n$.

Therefore, f is injective. We now show that f is surjective by case analysis on the sign of some integer t in **Z**.

- Let t be positive, then t will appear in an even position in the sequence, thus $f(2k)=2k/2=t$ with $t=k$. This implies that for every positive value t in **Z** there is a natural number 2k.
- Let t be negative, then t will appear in an odd position in the sequence, thus $f(2k-1)=-(2k-1-1)/2=t$ with $t=-k$. This implies that for every negative value t in **Z** there is a natural number 2k-1.

Therefore, f is surjective. (QED)

Strings

Example 4: Show that the set of finite strings S over the lowercase letters is countably infinite.

Proof: Show that the strings can be listed in a sequence. First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
3. Then all the strings of length 2 in lexicographic order.
4. And so on.

This implies a bijection from $\mathbf{N}$ to S and hence it is a countably infinite set. ◀

The set of all Java programs is countable.

Example 5: Show that the set of all Java programs is countable.

Solution: Let *S* be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:

- Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
- If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
- We move on to the next string.

In this way we construct an implied bijection from **N** to the set of Java programs. Hence, the set of Java programs is countable. ◀

The Real Numbers are Uncountable

Example: Show that the real numbers are not countable.

Proof: It is sufficient to show that the real numbers between 0 and 1 are not countable. Proof by contradiction. Assume that the real numbers between 0 and 1 are countable, then we can list them,

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}\ldots d_{1n}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}\ldots d_{2n}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}\ldots d_{3n}\ldots \\ &\vdots \\ r_n &= 0.d_{n1}d_{n2}d_{n3}\ldots d_{nn}\ldots \\ &\vdots \end{aligned}$$

However, given this list we can now construct a new real number r_q between 0 and 1 that does not appear on this list,

$$r_q = 0.d_{q1}d_{q2}d_{q3}\ldots d_{qn}\ldots$$

$$\text{with } d_{q1} \neq d_{11}, d_{q2} \neq d_{22}, d_{q3} \neq d_{33}, \ldots, d_{qn} \neq d_{nn}, \ldots$$

The Real Numbers are Uncountable

With this construction r_q differs from any real number on the list in at least one position. This is a contradiction. Therefore, the real numbers between 0 and 1 are uncountable. (QED)