Mathematical Induction

Section 5.1

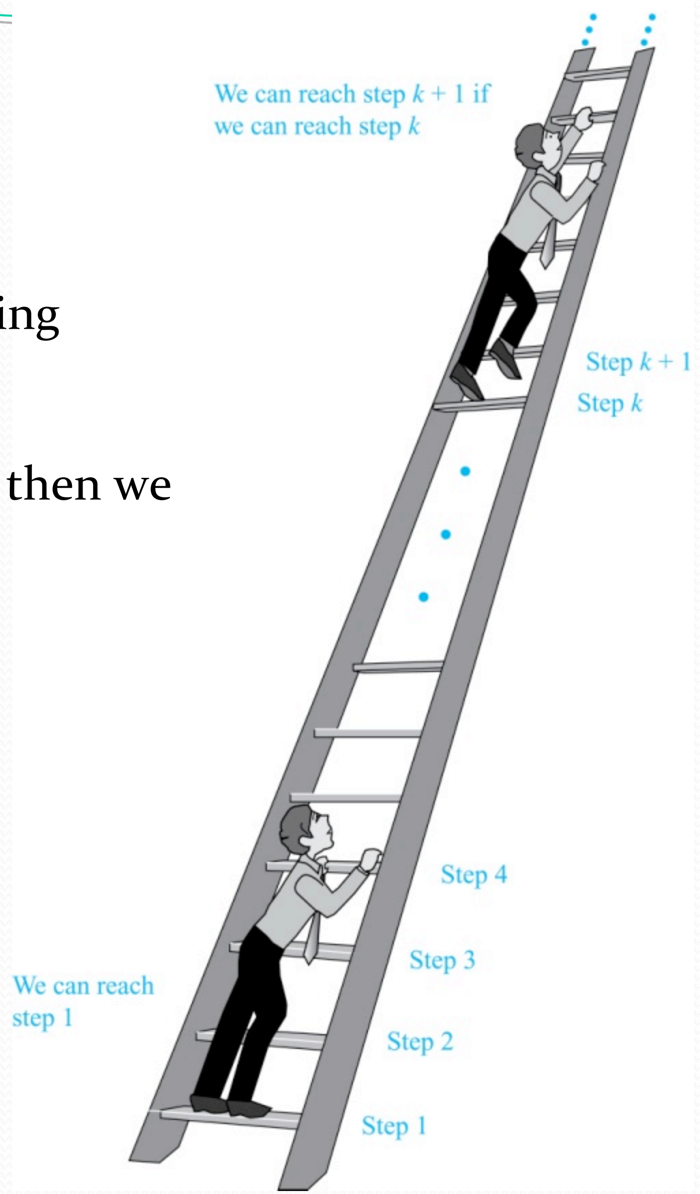
Climbing an Infinite Ladder

Suppose we have an infinite ladder and the following capabilities:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Then,

From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.



This example motivates proof by mathematical induction.

Principle of Mathematical Induction

Principle of Mathematical Induction: To prove that $P(n)$ is true for *all* positive integers n, we complete these steps:

- *Basis Step*: Show that $P(1)$ is true.
- *Inductive Step*: Show that $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

To complete the inductive step, we assume the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k and show that $P(k+1)$ is true, which then makes the implication true.

Climbing an Infinite Ladder Example:

- BASIS STEP: By (1), we can reach rung 1.
- INDUCTIVE STEP: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung $k+1$.

Hence, $P(k) \rightarrow P(k+1)$ is true for all positive integers k. We can reach every rung on the ladder.

Important Points About Using Mathematical Induction

- Mathematical induction can be expressed as the rule of inference

$$[P(1) \land \forall k\, [P(k) \rightarrow P(k+1)]] \rightarrow \forall n\, P(n),$$

where the domain is the set of positive integers.

- In a proof by mathematical induction, we <u>do not</u> assume that $P(k)$ is true for all positive integers!
- But, we show that if we assume that $P(k)$ is true, then $P(k+1)$ must also be true.

Proving a Summation Formula by Mathematical Induction

Example: Show that: $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof: Proof by induction.

- BASIS STEP: $P(1)$ is true since $1(1 + 1)/2 = 1$.
- INDUCTIVE STEP: Assume true for $P(k)$. The inductive hypothesis is
$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Under this assumption we show that P(k+1) is also true,

$$\begin{aligned} 1+2+\ldots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \quad \text{(by inductive hypothesis)} \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)\big((k+1)+1\big)}{2} \end{aligned}$$

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Conjecturing and Proving Correct a Summation Formula

Example: Conjecture and prove correct a formula for the sum of the first n positive odd integers. Then prove your conjecture.

Solution: We have: $1 = 1$, $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$, $1 + 3 + 5 + 7 + 9 = 25$.

- We can conjecture that the sum of the first n positive odd integers is n^2,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

- We prove the conjecture correct with mathematical induction.
- BASIS STEP: $P(1)$ is true since $(2(1)-1)=1=1^2$.
- INDUCTIVE STEP: $P(k) \to P(k+1)$ for every positive integer k.

  Assume the inductive hypothesis $P(k)$ holds and then show that $P(k+1)$ holds has well.

Inductive Hypothesis: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

- So, assuming $P(k)$, it follows that:

$$\begin{aligned} 1+3+5+\ldots+(2k-1)+\big(2(k+1)-1\big) &= 1+3+5+\ldots+(2k-1)+(2k+1) \\ &= k^2+(2k+1) \quad \text{(by inductive hypothesis)} \\ &= k^2+2k+1 \\ &= (k+1)^2 \end{aligned}$$

- Hence, we have shown that $P(k+1)$ follows from $P(k)$. Therefore the sum of the first n positive odd integers is n^2.

Proving Inequalities

Example: Use mathematical induction to prove that $n < 2^n$ for all positive integers n.

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

- BASIS STEP: $P(1)$ is true since $1 < 2^1 = 2$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer k. We now show that $P(k + 1)$ holds. Since by the inductive hypothesis, $k < 2^k$, it follows that:

$$\begin{aligned} k+1 &< 2^k + 1 \quad \text{(by the inductive hypothesis)} \\ &\leq 2^k + 2^k \quad \text{(using property } 1 \leq p^q \text{ for all } p,q \in N\text{)} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

Therefore $n < 2^n$ holds for all positive integers n. ◀

Proving Inequalities

Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

- BASIS STEP: $P(4)$ is true since $2^4 = 16 < 4! = 24$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$. To show that $P(k + 1)$ holds:

$$
\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&< 2 \cdot k! \qquad \textit{(by the inductive hypothesis)} \\
&< (k + 1)k! \quad (\text{ using } 2<(\text{k}+1) \text{ for } k \geq 4) \\
&= (k + 1)!
\end{aligned}
$$

Therefore, $2^n < n!$ holds, for every integer $n \geq 4$. ◀

Note: that here the basis step is $P(4)$, since $P(0)$, $P(1)$, $P(2)$, and $P(3)$ are all false.

Proving Divisibility Results

Example: Use mathematical induction to prove that $n^3 - n$ is divisible by 3, for every positive integer n.

Solution: Let $P(n)$ be the proposition that $n^3 - n$ is divisible by 3.

- BASIS STEP: $P(1)$ is true since $1^3 - 1 = 0$, which is divisible by 3.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k^3 - k$ is divisible by 3, for an arbitrary positive integer k. To show that $P(k + 1)$ follows:

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3(k^2 + k)\end{aligned}$$

By the inductive hypothesis, the first term $(k^3 - k)$ is divisible by 3 and the second term is divisible by 3 since it is an integer multiplied by 3.

Therefore, $n^3 - n$ is divisible by 3, for every integer positive integer n. ◀

Number of Subsets of a Finite Set

Example: Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

Solution: Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

- Basis Step: $P(0)$ is true, because the empty set has only itself as a subset and $2^0 = 1$.
- Inductive Step: Assume $P(k)$ is true for an arbitrary nonnegative integer k.

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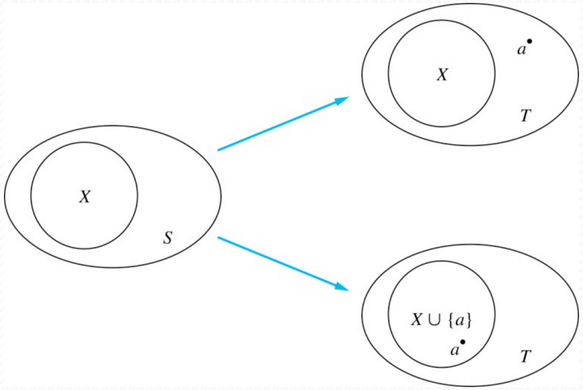
Number of Subsets of a Finite Set

Inductive Hypothesis: For an arbitrary nonnegative integer k, every set with k elements has 2^k subsets.

- Let T be a set with $k + 1$ elements. Then $T = S \cup \{a\}$, where $a \in T$ and $S = T - \{a\}$. Hence $|T| = k+1$.
- For each subset X of S, there are exactly two subsets of T, i.e., X and $X \cup \{a\}$.



- By the inductive hypothesis S has 2^k subsets. Since there are two subsets of T for each subset of S, the number of subsets of T is $2 \cdot 2^k = 2^{k+1}$.

Validity of Mathematical Induction

- Mathematical induction is valid (holds for all positive integers) because of the well ordering property, which states that every nonempty subset of the set of positive integers has a least element.
- Here is a proof by contradiction:
  - Suppose that $P(1)$ holds and $P(k) \rightarrow P(k+1)$ is true for all positive integers k.
  - Assume there is at least one positive integer n for which $P(n)$ is false. Then the set S of positive integers for which $P(n)$ is false is nonempty.
  - By the well-ordering property, S has a least element, say m.
  - We know that m cannot be 1 since $P(1)$ holds.
  - Since m is positive and greater than 1, $m - 1$ must be a positive integer. Since $m - 1 < m$, it is not in S, so $P(m - 1)$ must be true.
  - But then, since the conditional $P(k) \rightarrow P(k+1)$ for every positive integer k holds, $P(m)$ must also be true. This contradicts $P(m)$ being false.
  - Hence, $P(n)$ must be true for every positive integer n.