

# The Basics of Counting

Section 6.1

# Arrangements of Objects

- The study of arrangements of objects is called combinatorics
- Counting objects in these arrangements is an important part of combinatorics as well as in other disciplines such as computer science, biology, and statistics.
- We start with basic techniques of counting objects.

# Basic Counting Principles: The Product Rule

**The Product Rule:** Assume a procedure can be broken down into a sequence of two tasks and there are  $n_1$  ways to do the first task and  $n_2$  ways to do the second task. Then there are  $n_1 \cdot n_2$  ways to do the procedure.

# Basic Counting Principles: The Product Rule

**Example:** How many bit strings of length eight are there?

**Solution:** Since each of the eight bits is either a 0 or a 1, the answer is  $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8 = 256$ .

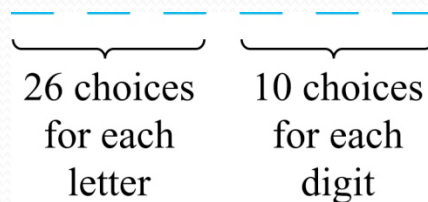
Let eight bits be a byte and assign an unsigned integer value between 0 and 255 to each of the 256 bit strings, then a byte can represent 256 unique unsigned integers between 0 and 255.

# The Product Rule

**Example:** How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

**Solution:** By the product rule,

there are  $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$  different possible license plates.



# Counting Functions

**Counting Functions:** How many functions are there from a set with  $m$  elements to a set with  $n$  elements?

**Solution:** Since a function represents a choice of one of the  $n$  elements of the codomain for each of the  $m$  elements in the domain, the product rule tells us that there are  $n \cdot n \cdots n = n^m$  such functions.

# Counting Functions

**Counting One-to-One Functions:** How many one-to-one functions are there from a set with  $m$  elements to one with  $n$  elements?

**Solution:** Assume  $m \leq n$ . Suppose the elements in the domain are  $a_1, a_2, \dots, a_m$ . Then there are  $n$  ways to choose the value of a function at  $a_1$  and  $n-1$  ways to choose the value of the function at  $a_2$ , etc. The product rule tells us that there are

$$n(n-1)(n-2)\cdots(n-(m-1))$$
 such functions.

# Telephone Numbering Plan

**Example:** The *North American numbering plan (NANP)* specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let  $X$  denote a digit from 0 through 9.
- Let  $N$  denote a digit from 2 through 9.
- Let  $Y$  denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was  $NYX-NNX-XXX$ .
- In the new plan, the format is  $NXX-NXX-XXX$ .

How many different telephone numbers are possible under the old plan and the new plan?

**Solution:** Use the Product Rule.

- There are  $8 \cdot 2 \cdot 10 = 160$  area codes with the format  $NYX$ .
- There are  $8 \cdot 10 \cdot 10 = 800$  area codes with the format  $NXX$ .
- There are  $8 \cdot 8 \cdot 10 = 640$  office codes with the format  $NNX$ .
- There are  $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$  station codes with the format  $XXXX$ .

Number of old plan telephone numbers:  $160 \cdot 640 \cdot 10,000 = 1,024,000,000$ .

Number of new plan telephone numbers:  $800 \cdot 800 \cdot 10,000 = 6,400,000,000$ .



# Counting Subsets of a Finite Set

**Counting Subsets of a Finite Set:** Use the product rule to show that the number of different subsets of a finite set  $S$  is  $2^{|S|}$ .

**Solution:** When the elements of  $S$  are listed in an arbitrary order, there is a one-to-one correspondence between subsets of  $S$  and bit strings of length  $|S|$ . When the  $i$ th element is in the subset, the bit string has a 1 in the  $i$ th position and a 0 otherwise.

By the product rule, there are  $2^{|S|}$  such bit strings, and therefore  $2^{|S|}$  subsets.

# Product Rule in Terms of Sets

- If  $A_1, A_2, \dots, A_m$  are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product  $A_1 \times A_2 \times \dots \times A_m$  is done by choosing an element in  $A_1$ , an element in  $A_2$ , ..., and an element in  $A_m$ .
- By the product rule, it follows that:  
$$|A_1 \times A_2 \times \dots \times A_m| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_m|.$$

## Basic Counting Principles: The Sum Rule

**The Sum Rule:** If a task can be done either in one of  $n_1$  ways or in one of  $n_2$  ways, where none of the set of  $n_1$  ways is the same as any of the  $n_2$  ways, then there are  $n_1 + n_2$  ways to do the task.

## Basic Counting Principles: The Sum Rule

**Example:** The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

**Solution:** By the sum rule it follows that there are  $37 + 83 = 120$  possible ways to pick a representative.

# The Sum Rule in terms of sets.

- The sum rule can be phrased in terms of sets.

$|A \cup B| = |A| + |B|$  as long as  $A$  and  $B$  are disjoint sets.

- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when  $A_i \cap A_j = \emptyset$  for all  $i, j$ .

# Combining the Sum and Product Rule

**Example:** Suppose statement labels in a programming language can be either a single uppercase letter or a lowercase letter followed by a digit. Find the number of possible labels.

**Solution:** Use the sum and the product rule.

$$26 + 26 \cdot 10 = 286$$

# Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems.  
**Example:** Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

**Solution:** Let  $P$  be the total number of passwords, and let  $P_6$ ,  $P_7$ , and  $P_8$  be the passwords of length 6, 7, and 8.

- By the sum rule  $P = P_6 + P_7 + P_8$ .
- To find each of  $P_6$ ,  $P_7$ , and  $P_8$ , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$$P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560.$$

$$P_7 = 36^7 - 26^7 =$$

$$78,364,164,096 - 8,031,810,176 = 70,332,353,920.$$

$$P_8 = 36^8 - 26^8 =$$

$$2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880.$$

Consequently,  $P = P_6 + P_7 + P_8 = 2,684,483,063,360$ .

# Basic Counting Principles: Subtraction Rule

**Subtraction Rule:** If a task can be done either in one of  $n_1$  ways or in one of  $n_2$  ways, then the total number of ways to do the task is  $n_1 + n_2$  minus the number of ways to do the task that are common to the two different ways.

- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

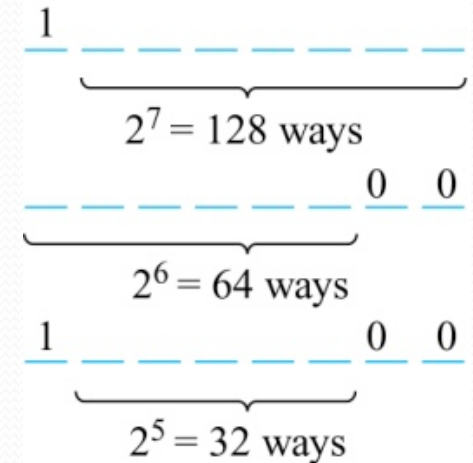


# Counting Bit Strings

**Example:** How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

**Solution:** Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit:  $2^7 = 128$
- Number of bit strings of length eight that end with bits 00:  $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 :  $2^5 = 32$



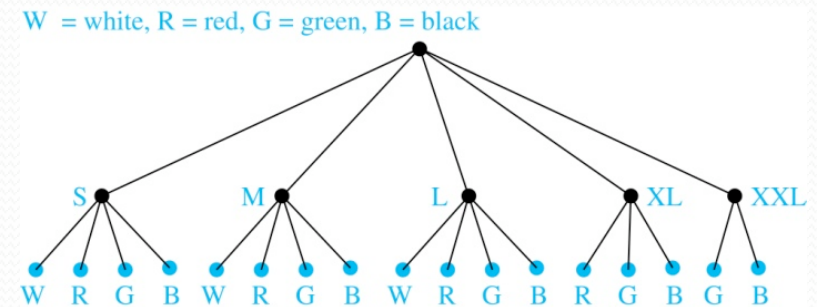
Hence, the number is  $128 + 64 - 32 = 160$ .

# Tree Diagrams

- **Tree Diagrams:** We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.

# Tree Diagrams

- **Example:** Suppose that “I Love Discrete Math” T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of stores that the campus book store needs to stock to have one of each size and color available?
- **Solution:** Draw the tree diagram.



- The store must stock 17 T-shirts.