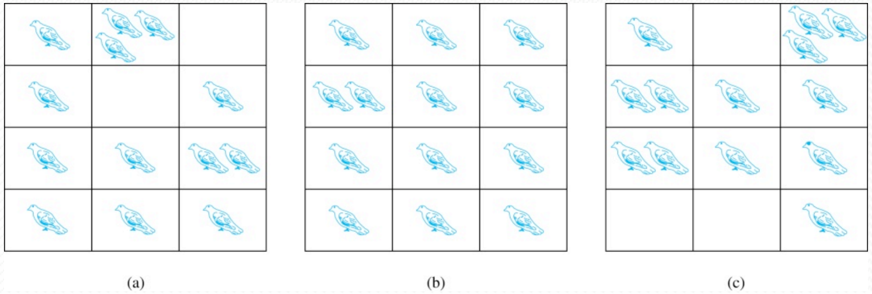
The Pigeonhole Principle

Section 6.2

The Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



- It is an important proof principle

The Pigeonhole Principle (more formal)

Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects.

◀

The Pigeonhole Principle

Theorem: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the codomain of f .
- Put in the box for y all of the elements x from the domain such that $f(x) = y$.
- Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one. ◀

Pigeonhole Principle

Example: In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

Pigeonhole Principle

- **Example**: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

- **Solution**: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Generalized Pigeonhole Principle

- The pigeonhole principle states that there must be at least two objects in the same box when there are more objects than boxes. However, even more can be said when the number of objects exceeds a *multiple* of the number of boxes.

The Generalized Pigeonhole Principle

- **Example**: Given a group of 100 people, at minimum, how many people were born in the same month?

- **Solution:** We have 12 months, so 100/12 = 8.33 gives us the content of each month. But we cannot have partial people in each month, so we round up:

$$\left\lceil \frac{100}{12} \right\rceil = 9$$

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

The Generalized Pigeonhole Principle

Proof: We use a proof by contradiction. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right)$$

but

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < N/k + 1$ has been used. This is a contradiction because there are a total of N objects. ◀

The Generalized Pigeonhole Principle

Example: Among any set of 21 decimal *digits* there must be 3 that are the same.

Solution: This follows because when 21 objects are distributed into 10 boxes, one box must have

$$\left\lceil \frac{21}{10} \right\rceil = \lceil 2.1 \rceil = 3$$

elements.

The Generalized Pigeonhole Principle

- A common type of problem asks for the minimum number of objects such that at least r of these objects must be in one of k boxes when these objects are distributed among the boxes.
- When we have N objects, the generalized pigeonhole principle tells us there must be at least r objects in one of the boxes as long as $\lceil N/k \rceil \geq r$.
- The smallest integer N with $N/k > r - 1$, namely, $N = k(r - 1) + 1$, is the smallest integer satisfying the inequality $\lceil N/k \rceil \geq r$.

The Generalized Pigeonhole Principle

- **Example**: What is the minimum number of students required in a course to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

- **Solution**: We have k = 5 and r = 6. We need to compute N such that $\lceil N/k \rceil = r$ or more precisely $\lceil N/5 \rceil = 6$. We can compute the smallest integer with this property as N = k(r – 1) + 1. Plugging k and r into this equation gives us N = 5(6-1)+1 = 26. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

The Generalized Pigeonhole Principle

- **Example**: What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

- **Solution**: There are eight million different phone numbers of the form NXX-XXXX (as we have shown previously). Hence, by the generalized pigeonhole principle, among 25 million telephones, at least $\lceil 25{,}000{,}000/8{,}000{,}000 \rceil = 4$ of them must have identical phone numbers. Hence, at least four area codes are required to ensure that all 10-digit numbers are different.