

Relations and Their Properties

Section 9.1

Relationships

- Relationships between elements of sets occur in many contexts. Every day we deal with relationships such as those between
 - a business and its telephone number,
 - an employee and his or her salary,
 - a person and a relative,
 - and so on.
- In mathematics we study relationships such as those between
 - a positive integer and one that it divides,
 - a real number and one that is larger than it,
 - a real number x and the value $f(x)$ where f is a function,
 - and so on.
- In computer science we study relationships such as that between a
 - program and a variable it uses,
 - a computer language and a valid statement in this language

Relations

- Relationships between elements of sets are represented using a structure called a *relation*, which is just a subset of the Cartesian product of the sets.
- Relations can be used to solve problems such as
 - determining which pairs of cities are linked by airline flights in a network,
 - finding a viable order for the different phases of a complicated project,
 - producing a useful way to store information in computer databases.

Relations

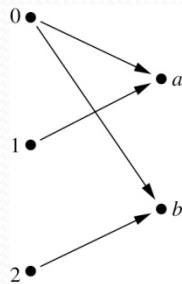
- The most direct way to express a relationship between elements of *two* sets is to use *ordered pairs* made up of two related elements.
- For this reason, sets of ordered pairs are called *binary relations*.

Binary Relations

Definition: A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a), (2, b)\}$ is a relation from A to B .
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A .

Binary Relation on a Set

Definition: A binary relation R on a set A is a subset of $A \times A$ or a relation from A to A .

Example:

- Suppose that $A = \{a, b, c\}$. Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on A .
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ are $(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3),$ and $(4, 4)$.

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A ?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A .

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_5 = \{(a,b) \mid a = b + 1\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

$(1,1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Solution: Checking the conditions that define each relation, we see that the pair $(1,1)$ is in R_1 , R_3 , R_4 , and R_6 ; $(1,2)$ is in R_1 and R_6 ; $(2,1)$ is in R_2 , R_5 , and R_6 ; $(1, -1)$ is in R_2 , R_3 , and R_6 ; $(2,2)$ is in R_1 , R_3 , and R_4 .

Reflexive Relations

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall x[x \in U \rightarrow (x,x) \in R]$$

Example: The following relations on the integers are reflexive:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\}.$$

The following relations are not reflexive:

$$R_2 = \{(a,b) \mid a > b\} \text{ (note that } 3 \not> 3\text{),}$$

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that } 3 \neq 3 + 1\text{),}$$

$$R_6 = \{(a,b) \mid a + b \leq 3\} \text{ (note that } 4 + 4 \not\leq 3\text{).}$$

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y [(x,y) \in R \rightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

The following are not symmetric:

$$R_1 = \{(a,b) \mid a \leq b\} \text{ (note that } 3 \leq 4, \text{ but } 4 \not\leq 3),$$

$$R_2 = \{(a,b) \mid a > b\} \text{ (note that } 4 > 3, \text{ but } 3 \not> 4),$$

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that } 4 = 3 + 1, \text{ but } 3 \neq 4 + 1).$$

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.
Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y [(x, y) \in R \wedge (y, x) \in R \rightarrow x = y]$$

- **Example:** The following relations on the integers are antisymmetric:

$$R_1 = \{(a, b) \mid a \leq b\},$$

$$R_2 = \{(a, b) \mid a > b\},$$

$$R_4 = \{(a, b) \mid a = b\},$$

$$R_5 = \{(a, b) \mid a = b + 1\}.$$

For any integer, if $a \leq b$ and $a \geq b$, then $a = b$.

The following relations are not antisymmetric:

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

(note that both $(1, -1)$ and $(-1, 1)$ belong to R_3),

$$R_6 = \{(a, b) \mid a + b \leq 3\} \text{ (note that both } (1, 2) \text{ and } (2, 1) \text{ belong to } R_6).$$

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z [(x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R]$$

- **Example:** The following relations on the integers are transitive:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\}.$$

For every integer, $a \leq b$
and $b \leq c$, then $b \leq c$.

The following are not transitive:

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that both } (3,2) \text{ and } (4,3) \text{ belong to } R_5, \text{ but not } (3,3)\text{),}$$

$$R_6 = \{(a,b) \mid a + b \leq 3\} \text{ (note that both } (2,1) \text{ and } (1,2) \text{ belong to } R_6, \text{ but not } (2,2)\text{).}$$

Examples

Consider the following relations on the set $\{1,2,3,4\}$,

- $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
- $R_2 = \{(1, 1), (1, 2), (2, 1)\}$,
- $R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
- $R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
- $R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
- $R_6 = \{(3, 4)\}$.

Reflexive: $\forall x[x \in U \rightarrow (x,x) \in R]$

Symmetric: $\forall x \forall y [(x,y) \in R \rightarrow (y,x) \in R]$

Antisymmetric: $\forall x \forall y [(x,y) \in R \wedge (y,x) \in R \rightarrow x = y]$

Transitive: $\forall x \forall y \forall z [(x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R]$

Combining Relations

- Given two relations R_1 and R_2 , we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example:** Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \quad R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

Composition

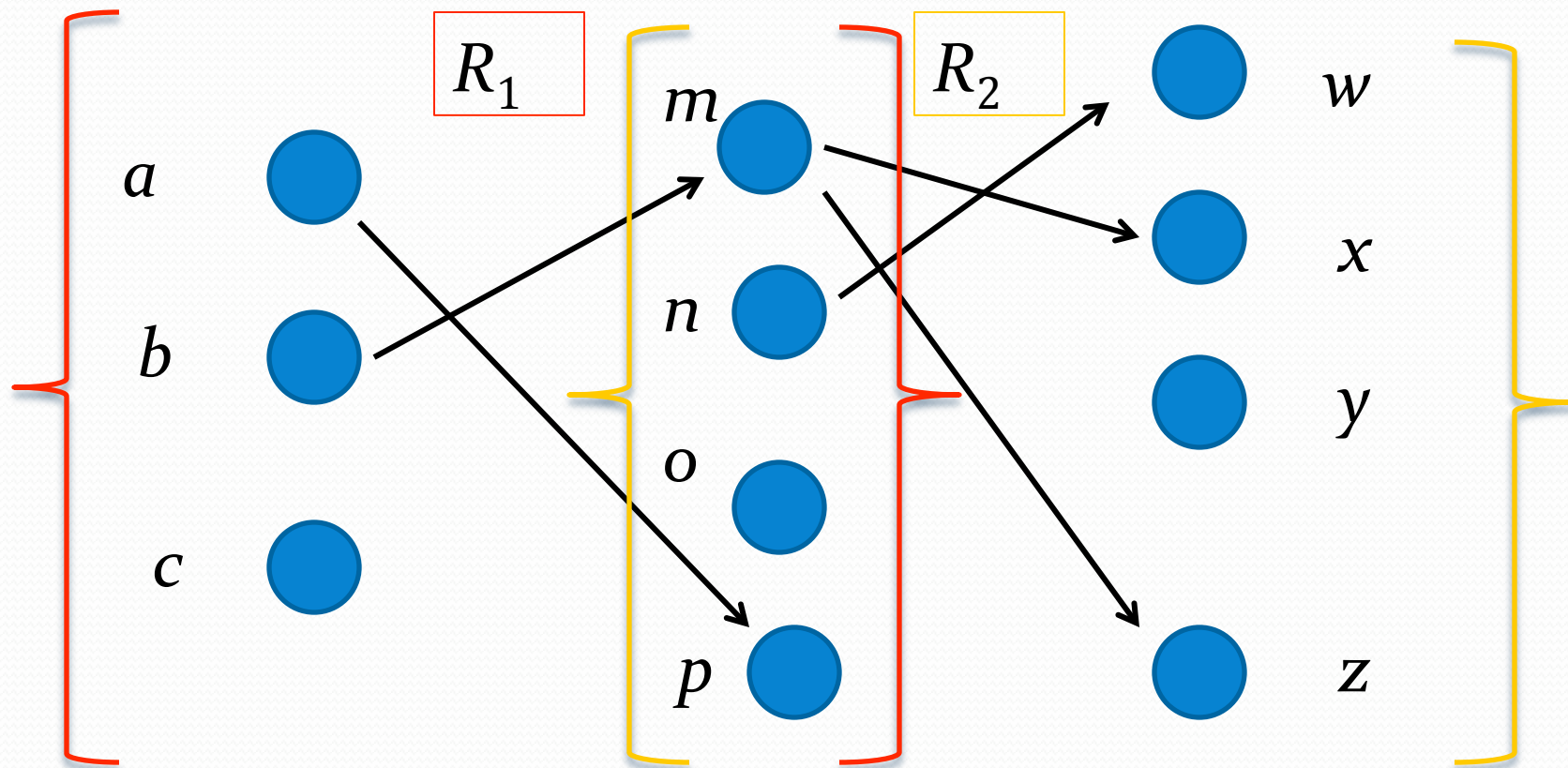
Definition: Suppose

- R_1 is a relation from a set A to a set B .
- R_2 is a relation from B to a set C .

Then the *composition* (or *composite*) of R_2 with R_1 , is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2 , then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$$R_1 \circ R_2 = \{(b, D), (b, B)\}$$