Equivalence Relations

Section 9.5

Equivalence Relations

Definition: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Equivalence Relations

Definition: Two elements a, and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Proof: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and therefore bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(c)$ also holds and therefore aRc. (QED)

Relation on Real Numbers

Example: Let R be the relation on the set of real numbers such that aRb if and only if a – b is an integer. Is R an equivalence relation?

Proof: Because a – a = 0 is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive. Now suppose that aRb. Then a – b is an integer, so b – a is also an integer. Hence, bRa. It follows that R is symmetric. If aRb and bRc, then a – b and b – c are integers. Therefore, a – c = (a – b) + (b – c) is also an integer. Hence, aRc. Thus, R is transitive. Consequently, R is an equivalence relation. (QED)

Bit String Equivalences

- **Example**: Let n be a positive integer and S a set of strings. Suppose that Rn is the relation on S such that sRn t ifand only if s = t, or both s and t have at least n characters and the first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t begins with the n characters at the start of s. For example, let n = 3 and let S be the set of all bit strings. Then sR3 t either when s = t or both s and t are bit strings of length 3 or more that begin with the same three bits. For instance, 01 R3 01 and 00111 R3 00101, but 01 R 3 010 is not and 01011 R 3 01110 is not.
- Show that for every set S of strings and every positive integer n, R_n is an equivalence relation on S.

Bit String Equivalences

- **Solution**: We show that the relation Rn is reflexive, symmetric, and transitive.
  - *Reflexive*: The relation Rn is reflexive because s = s, so that sRn s whenever s is a string in S.
  - *Symmetric*: If sRn t, then either s = t or s and t are both at least n characters long that begin with the same n characters. This means that tRn s. We conclude that Rn is symmetric.
  - *Transitive*: Now suppose that sRn t and tRn u. Then either s = t or s and t are at least n characters long and s and t begin with the same n characters, and either t = u or t and u are at least n characters long and t and u begin with the same n characters. From this, we can deduce that either s = u or both s and u are n characters long and s and u begin with the same n characters, i.e. s Rn u. Consequently, Rn is transitive.
- It follows that Rn is an equivalence relation. (QED)

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not symmetric. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: *a* divides *a* for all *a*.
- *Not Symmetric*: For example, 2 divides 4, but 4 divides 2 does not hold. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that *a* divides *b* and *b* divides *c*. Then there are positive integers *k* and *l* such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so *a* divides *c*. Therefore, the relation is transitive.

Equivalence Classes

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s \mid (a,s) \in R\}$.

- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.

Bit String Equivalence Class

- **Example**: What is the equivalence class of the string 0111 with respect to the equivalence relation R3 from the Bit String Equivalence example on the set of all bit strings? (Recall that sR3 t if and only if s and t are bit strings with s = t or s and t are strings of at least three bits that start with the same three bits.)

- **Solution**: The bit strings equivalent to 0111 are the bit strings with at least three bits that begin with 011. These are the bit strings 011, 0110, 0111, 01100, 01101, 01110, 01111, and so on. Consequently,

$$[011]_{R_3} = \{011, 0110, 0111, 01100, 01101, 01110, 01111, \ldots\}.$$

Equivalence Classes

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) aRb

(*ii*) $[a] = [b]$

(*iii*) $[a] \cap [b] \neq \emptyset$

Equivalence Classes

Proof:

(a) We show that (i) implies (ii). Assume that aRb. Now suppose that $c \in [a]$. Then aRc. Because aRb and R is symmetric, bRa. Because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

(b) We show that (ii) implies (iii). Assume that [a] = [b]. It follows that $[a] \cap [b] \neq \varnothing$ since [a] is nonempty; $a \in [a]$ because R is reflexive.

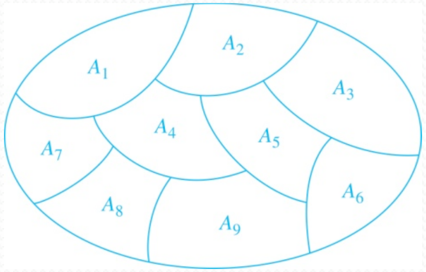
(c) We show that (iii) implies (i). Suppose that $[a] \cap [b] \neq \varnothing$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, aRc and bRc. By the symmetric property, cRb. Then by transitivity, because a c and cRb, we have aRb.

Because (i) implies (ii), (ii) implies (iii), and (iii) implies (i), the three statements, (i), (ii), and (iii), are equivalent.

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S.$



A Partition of a Set

An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,
$$\bigcup_{a \in A} [a]_R = A.$$

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- From Theorem 1, it follows that these equivalence classes are either equal ($[a] = [b]$ with $[a] \cap [b] \neq \emptyset$) or disjoint ($[a]_R \neq [b]_R$ with $[a]_R \cap [b]_R = \emptyset$).
- Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

An Equivalence Relation Partitions a Set

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, and so is $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition. (QED)