

Partial Orderings

Section 9.6

Partial Orderings

Definition 1: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, *antisymmetric*, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R) . Members of S are called *elements* of the poset.

Partial Orderings (*continued*)

Example: Show that the “greater than or equal” relation (\geq) is a partial ordering on the set of integers.

- *Reflexivity:* $a \geq a$ for every integer a .
- *Antisymmetry:* If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity:* If $a \geq b$ and $b \geq c$, then $a \geq c$.

Partial Orderings (*continued*)

Example: Show that the inclusion relation (\subseteq) is a partial ordering on the power set of a set S .

- *Reflexivity:* $A \subseteq A$ whenever A is a subset of S .
- *Antisymmetry:* If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity:* If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Partial Orderings (*continued*)

Example: Show that the divisibility relation ($|$) is a partial ordering on the set of integers.

- *Reflexivity:* $a | a$ for all integers a . (see Example 9 in Section 9.1)
- *Antisymmetry:* If a and b are positive integers with $a | b$ and $b | a$, then $a = b$. (see Example 12 in Section 9.1)
- *Transitivity:* Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c . Therefore, the relation is transitive.
- $(\mathbb{Z}^+, |)$ is a poset.

Comparability

Definition 2: Let \preceq denote any reflexive, antisymmetric, transitive relation, then the elements a and b of a poset (S, \preceq) are *comparable* if either $a \preceq b$ or $b \preceq a$. When a and b are elements of S so that neither $a \preceq b$ nor $b \preceq a$, then a and b are called *incomparable*.

Definition 3: If (S, \preceq) is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and \preceq is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

Definition 4: (S, \preceq) is a well-ordered set if it is a poset such that \preceq is a total ordering and every nonempty subset of S has a least element.