# Chapter 6:
# NFA Applications

# Implementing NFAs

- The problem with implementing NFAs is that, being nondeterministic, they define a more complex computational procedure for testing language membership.
- To implement an NFA we must give a computational procedure that can look at a string and decide whether the NFA has at *least one sequence* of legal transitions on that string leading to an accepting state.
- This seems to require searching through all legal sequences for the given input string—but how?

# Implementing NFAs

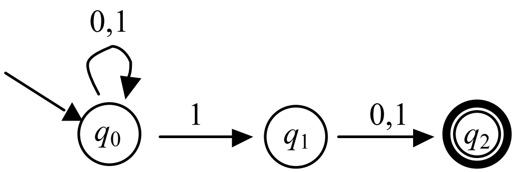
- One approach is to convert the NFA into a DFA and implement that instead.
- This NFA/DFA conversion is both useful and theoretically interesting: the fact that it is always possible shows that in spite of their extra flexibility, *NFAs have exactly the same power as DFAs*. They can define exactly the regular languages.

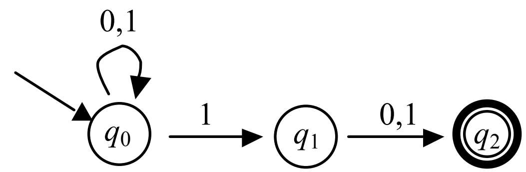
# Outline

- 6.1 NFA Implemented With Backtracking Search
- 6.2 NFA Implemented With Bit-Mapped Parallel Search
- 6.3 The Subset Construction
- 6.4 NFAs Are Exactly As Powerful As DFAs
- 6.5 DFA Or NFA?

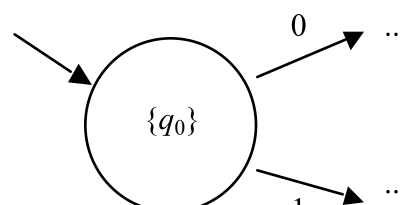
# From NFA To DFA

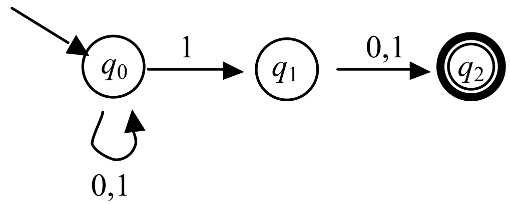
- For any NFA, there is a DFA that recognizes the same language
- Proof is by construction: a DFA that keeps track of the set of states the NFA might be in
- This is called the *subset construction*
- First, an example starting from this NFA:



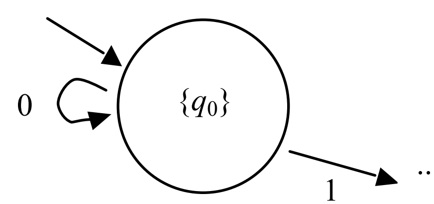


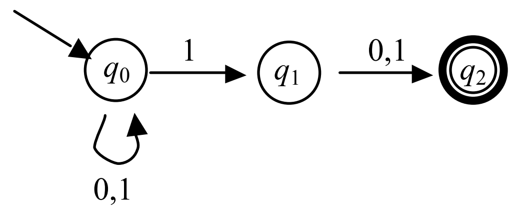

- Initially, the set of states the NFA could be in is just $\{q_0\}$
- So our DFA will keep track of that using a start state labeled $\{q_0\}$:



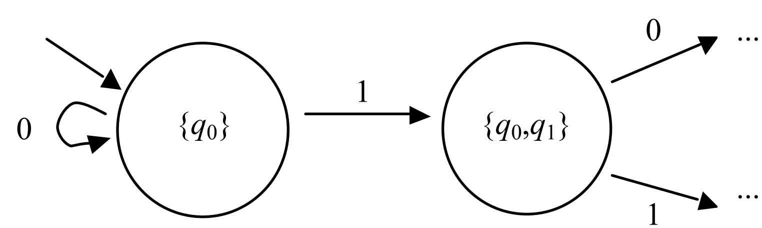


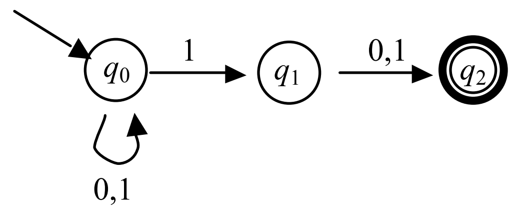

- Now suppose the set of states the NFA could be in is $\{q_0\}$, and it reads a 0
- The set of possible states after reading the 0 is $\{q_0\}$, so we can show that transition:



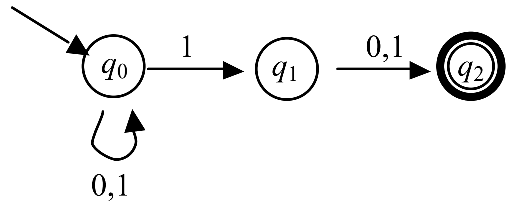


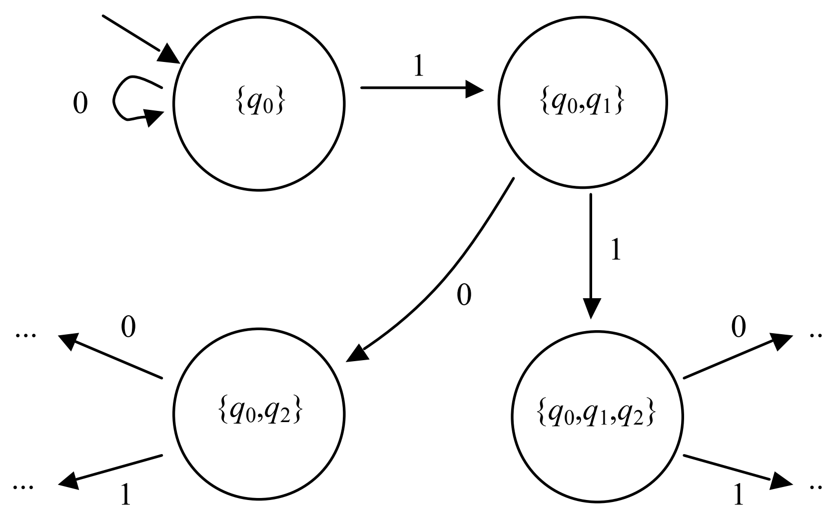

- Suppose the set of states the NFA could be in is $\{q_0\}$, and it reads a 1
- The set of possible states after reading the 1 is $\{q_0,q_1\}$, so we need another state:





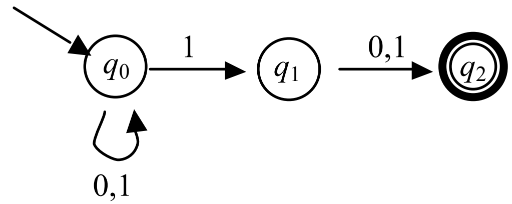

- From $\{q_0,q_1\}$ on a 0, the next set of possible states is $\delta(q_0,0) \cup \delta(q_1,0) = \{q_0,q_2\}$
- From $\{q_0,q_1\}$ on a 1, the next set of possible states is $\delta(q_0,1) \cup \delta(q_1,1) = \{q_0,q_1,q_2\}$
- Adding these transitions and states, we get…


$q_0$
1
$q_1$
0,1
$q_2$
0,1

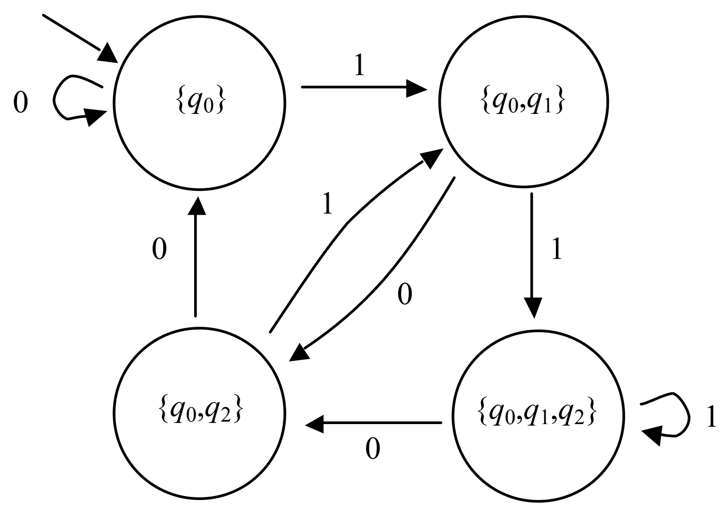

0
$\{q_0\}$
1
$\{q_0,q_1\}$
0
1
0
...
$\{q_0,q_2\}$
...
1
$\{q_0,q_1,q_2\}$
0
...
1
...

# And So On

- The DFA construction continues
- Eventually, we find that no further states are generated
- That's because there are only finitely many possible sets of states: $P(Q)$
- In our example, we have already found all sets of states reachable from $\{q_0\}$…



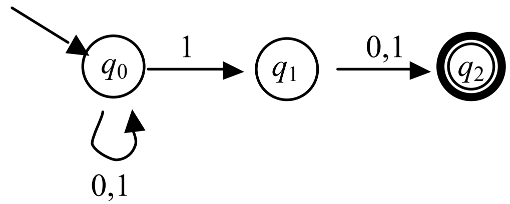
1
0,1
0,1
$q_0$
$q_1$
$q_2$

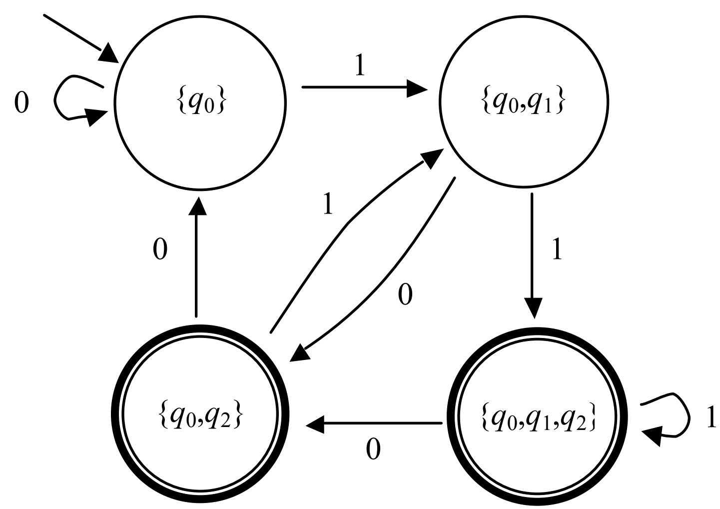
0
1
1
0
0
1
0
1
$\{q_0\}$
$\{q_0,q_1\}$
$\{q_0,q_2\}$
$\{q_0,q_1,q_2\}$

# Accepting States

- It only remains to choose the accepting states
- An NFA accepts *x* if its set of possible states after reading *x* includes at least one accepting state
- So our DFA should accept in all sets that contain at least one NFA accepting state



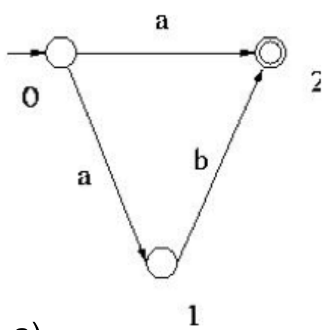
$q_0$
1
$q_1$
0,1
$q_2$
0,1

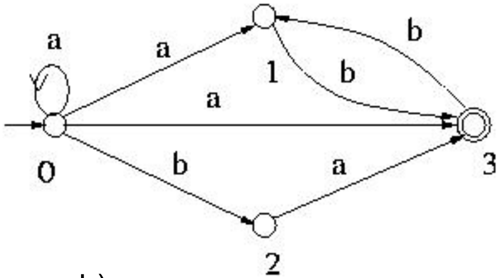
0
$\{q_0\}$
1
$\{q_0,q_1\}$
1
0
1
0
$\{q_0,q_2\}$
$\{q_0,q_1,q_2\}$
1
0

# Some Exercises

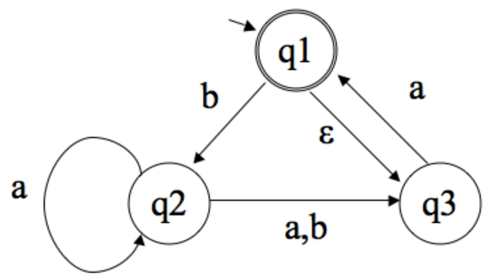
Convert the following NFAs into DFAs.




a)




b)




c)

# Implementation Note

- The subset construction defined the DFA transition function by

$$\delta_D(R,a) = \bigcup_{r \in R} \delta_N^*(r,a)$$

for some set of states R.

# Start State Note

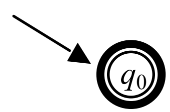
- In the subset construction, the start state for the new DFA is

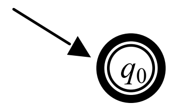
$$q_D = \delta_N^*(q_N, \varepsilon)$$

- Often this is the same as $q_D = \{q_N\}$, as in our earlier example
- But the difference is important if there are $\varepsilon$-transitions from the NFA's start state

# Empty-Set State Note

- The empty set is a subset of every set
- So the full subset construction always produces a DFA state for {}
- This is reachable from the start state if there is some string *x* for which the NFA has no legal sequence of moves: $\delta_N^*(q_N,x) = \{\}$
- For example, this NFA, with $L(N) = \{\varepsilon\}$





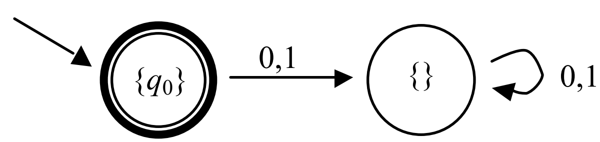

$$\delta_D\left(\left\{q_0\right\},0\right) = \bigcup_{r\in\left\{q_0\right\}} \delta_N^*\left(r,0\right) = \left\{\quad\right\}$$

$$\delta_D\left(\left\{q_0\right\},1\right) = \bigcup_{r\in\left\{q_0\right\}} \delta_N^*\left(r,1\right) = \left\{\quad\right\}$$

$$\delta_D\left(\left\{\quad\right\},0\right) = \bigcup_{r\in\left\{\quad\right\}} \delta_N^*\left(r,0\right) = \left\{\quad\right\}$$

$$\delta_D\left(\left\{\quad\right\},1\right) = \bigcup_{r\in\left\{\quad\right\}} \delta_N^*\left(r,1\right) = \left\{\quad\right\}$$

- $P(\{q_0\}) = \{ \{\}, \{q_0\} \}$
- A 2-state DFA



# Trap State Provided

- The subset construction always provides a state for {}
- And it is always the case that

$$\delta_D(\{\ \}, a) = \bigcup_{r \in \{\ \}} \delta_N^*(r, a) = \{\ \}$$

so the {} state always has transitions back to itself for every symbol *a* in the alphabet

- It is a non-accepting trap state

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# NFAs Are Exactly As Powerful As DFAs

- We want to show that NFAs and DFAs are equivalent.
- This means we want to show that for any NFA there is a DFA and for any DFA there is an NFA.

# Lemma 6.3

If $L(N)$ for some NFA $N$, then $L(N)$ is a regular language.

Proof: Every NFA $N$ gives rise to an equivalent DFA $D$ via the subset construction with $L(N) = L(D)$. Therefore $L(N)$ is regular.

# Lemma 6.4

If $L$ is any regular language, there is some NFA $N$ for which $L(N) = L$.

Proof:

- DFAs are just special NFAs that have never have a choice.
- To turn a DFA into an NFA all we have to do is modify the transition function from returning single states to sets of states:
  - Let $L$ be any regular language
  - By definition there must be some DFA $M = (Q, \Sigma, \delta, q_0, F)$ with $L(M) = L$
  - Define a new NFA $N = (Q, \Sigma, \delta', q_0, F)$, where $\delta'(q,a) = \{\delta(q,a)\}$ for all $q \in Q$ and $a \in \Sigma$, and $\delta'(q,\varepsilon) = \{\}$ for all $q \in Q$
  - Now $\delta'^*(q,x) = \{\delta^*(q,x)\}$, for all $q \in Q$ and $x \in \Sigma^*$
  - Thus $L(N) = L(M) = L$

# Theorem 6.4

A language $L$ is $L(N)$ for some NFA $N$
if and only if $L$ is a regular language.

Proof:

- Follows immediately from the previous lemmas