Chapter Sixteen: Turing Machines

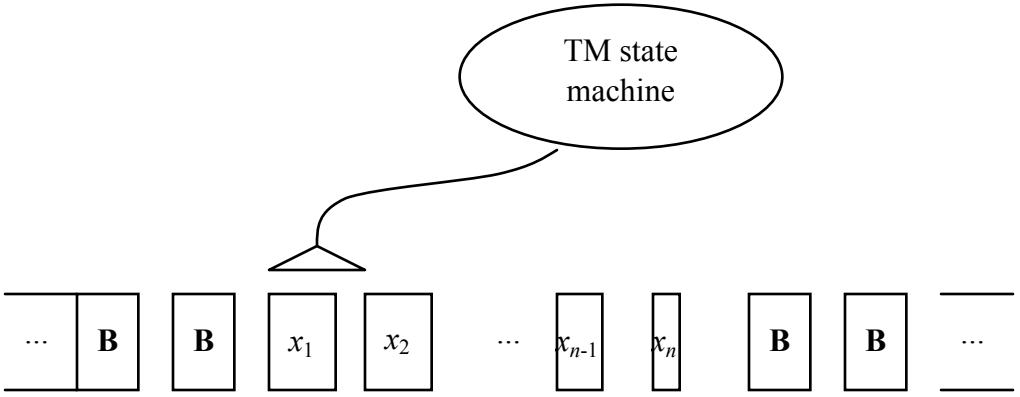
- We now turn to the most powerful kind of automaton we will study: the Turing machine.
  - Although it is only slightly more complicated than a finite-state machine, a Turing machine can do much more.
  - It is, in fact, so powerful no other, more powerful model exists.

- The TM is the strongest of computational mechanisms, it does have one weakness, revealed in this chapter: it can get stuck in an infinite loop.

Outline

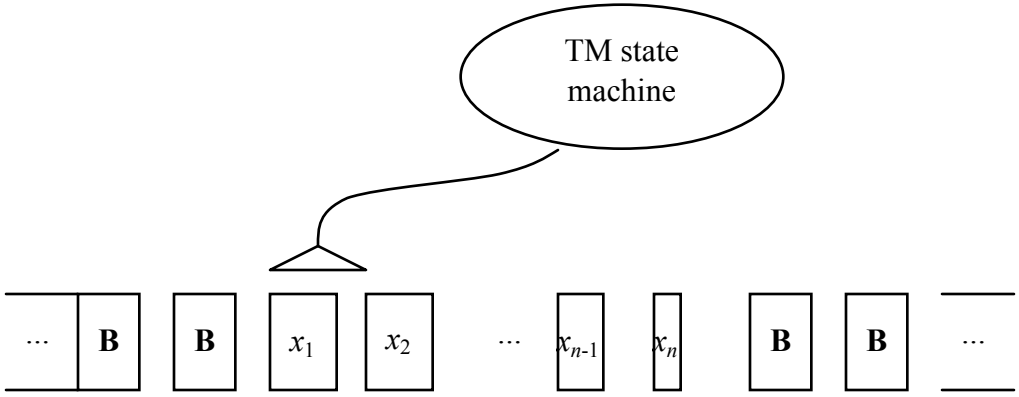
- 16.1 Turing Machine Basics
- 16.2 Simple TMs
- 16.3 A TM for $\{a^n b^n c^n\}$
- 16.4 The 7-Tuple
- 16.5 The Languages Defined By A TM
- 16.6 To Halt Or Not To Halt
- 16.7 A TM for $\{xcx \mid x \in \{a,b\}^*\}$
- 16.8 Three Tapes
- 16.9 Simulating DFAs
- 16.10 Simulating Other Automata

Turing Machine (TM)



- Input is a tape with one symbol at each position
- Tape extends infinitely in both directions
- Those positions not occupied by the input contain a special blank-cell symbol **B**
- The TM has a head that can read and write, and can move in both directions
- Head starts at the first input symbol

Turing Machine (TM)



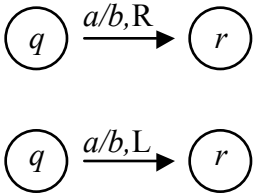
- A state machine controls the read/write head
- Move is determined by current state and symbol
- On each move: write a symbol, make a state transition, and move the head one place, left or right
- If it enters an accepting state, it halts and accepts

Difference From DFA

- TMs are like DFAs, but with:
  - Ability to move both left and right, unboundedly
  - Ability to write as well as read
- Important difference about how they accept:
  - A DFA reads to the end of the input, then accepts if that last state is accepting
  - A TM accepts the moment it enters an accepting state; final tape and head position don't matter; it doesn't even have to read all its input
  - Transitions leaving an accepting state are never used, so there is never any need for more than one accepting state

TM Transitions

- State-transition diagrams



- Right moves: if in state *q*, and the current tape symbol is *a*, write *b* over the *a*, move one place to the right, and go to state *r*
- Left moves: same, but move left

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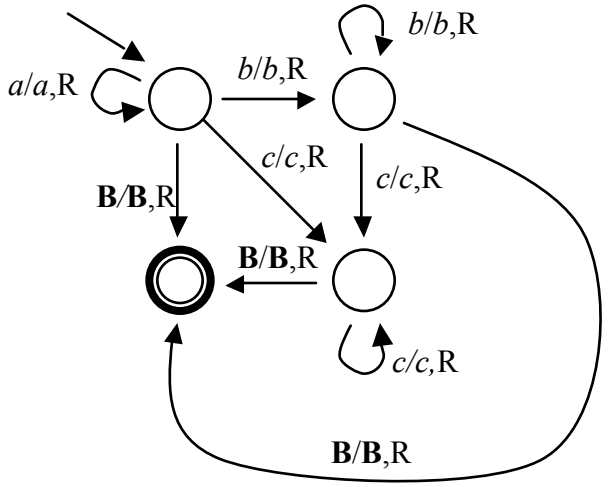
TMs For Regular Languages

- TMs can easily handle all regular languages
- In fact, a TM that always moves the head to the right works much like a DFA
- Except for that difference about the mechanism for accepting
- A TM only enters its accepting state when it has reached a final decision

L(*a***b***c**)



- Like a DFA:
  - Always moves right
  - Does not change the tape (always writes what it just read)
- Since it never moves left, it really doesn't matter what it writes
- It could write **B** on every move, erasing as it reads, and still accept the same language

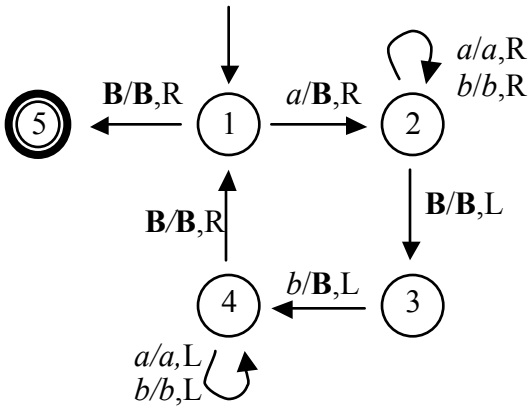
TMs For Context-Free Languages

- TMs can also easily handle all CFLs
- It is possible to take any stack machine and convert it into an equivalent TM that uses the infinite tape as a stack
- We'll demonstrate this more generally later
- But it is often easier to find some non-stack-oriented approach
- For example, $\{a^n b^n\}$…

Strategy For $\{a^n b^n\}$

- Repeatedly erase first (*a*) and last (*b*); if the string was in $\{a^n b^n\}$, this leaves nothing
- Five steps:
  1. If the current symbol is **B**, go to step 5. If the current symbol is *a*, write a **B** over it and go to step 2.
  2. Move right past any *a*s and *b*s. At the first **B**, move left one symbol and go to step 3.
  3. If the current symbol is *b*, write a **B** over it and go to step 4.
  4. Move left past any *a*s and *b*s. At the first **B**, move right one symbol and go to step 1.
  5. Accept.

$\{a^n b^n\}$



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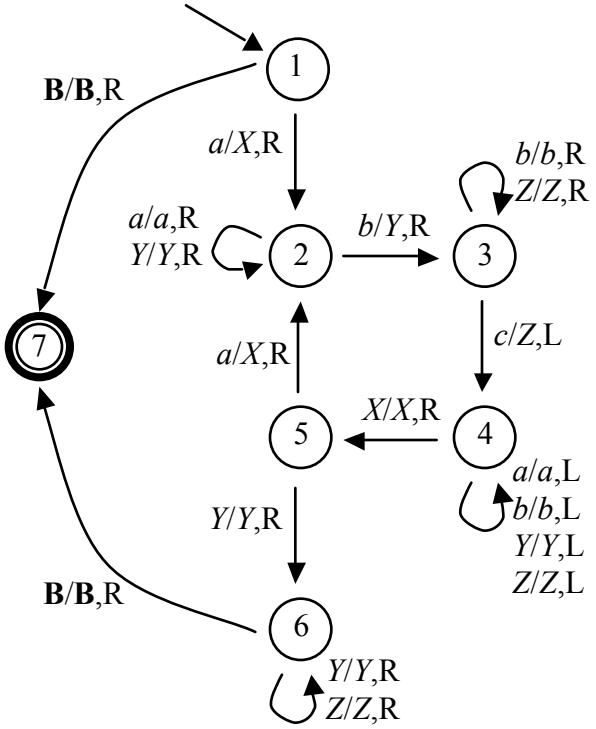
Strategy For $\{a^n b^n c^n\}$

1. If the current symbol is **B**, go to step 7. If the current symbol is *a*, write an *X* over it and go to step 2.
2. Move right past any *a*s and *Y*s. At the first *b*, write *Y* over it and go to step 3.
3. Move right past any *b*s and *Z*s. At the first *c*, write *Z* over it and go to step 4.
4. Move left past any *a*s, *b*s, *Z*s, and *Y*s. At the first *X*, move right 1 symbol and go to step 5.
5. If the current symbol is *a*, write *X* and go to step 2. If the current symbol is *Y* go to step 6.
6. Move right past any *Y*s and *Z*s. At the first **B**, go to step 7.
7. Accept.


1
2
3
4
5
6
7
B/B,R
a/X,R
a/a,R
Y/Y,R
b/Y,R
b/b,R
Z/Z,R
c/Z,L
X/X,R
a/X,R
a/a,L
b/b,L
Y/Y,L
Z/Z,L
Y/Y,R
B/B,R
Y/Y,R
Z/Z,R

Example

- See example slides

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The 7-Tuple

- A TM M is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, \mathbf{B}, q_0, F)$:
  - Q is the finite set of states
  - Σ is the input alphabet
  - Γ is the tape alphabet, with $\Sigma \subset \Gamma$ and $Q \cap \Gamma = \{\}$
  - $\delta \in (Q \times \Gamma \rightarrow Q \times \Gamma \times \{L,R\})$ is the transition function
  - **B** is the blank symbol, $\mathbf{B} \in \Gamma$, $\mathbf{B} \notin \Sigma$
  - $q_0 \in Q$ is the start state
  - $F \subseteq Q$ is the set of accepting states

The Transition Function

- $\delta \in (Q \times \Gamma \rightarrow Q \times \Gamma \times \{L,R\})$
- That is, $\delta(q,X) = (p,Y,D)$, where
  - Input q is the current state
  - Input X is the symbol at the current head position
  - Output p is the next state
  - Output Y is the symbol to write at the current head position (over the X)
  - Output D is a direction, L or R, to move the head
- δ is *deterministic*: at most one move from each configuration
- δ need not be defined over its whole domain, so there may be some q and X with no move $\delta(q,X)$

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TM Instantaneous Description

- An instantaneous description (ID) for a TM is a string *xqy*:
  - $x \in \Gamma^*$ represents the tape to the left of the head
  - *q* is the current state
  - $y \in \Gamma^*$ represents the tape at and to the right of the head
- ID strings are normalized as follows:
  - In *x*, leading **B**s are omitted, except when the left part is all **B**s, in which case *x* = **B**
  - In *y*, trailing **B**s are omitted, except when the right part is all **B**s, in which case *y* = **B**
  - We define a function *idfix*(*z*) that normalizes an ID string *z* in this way, removing (or adding) leading and trailing **B**s as necessary

A One-Move Relation On IDs

- We will write $I \mapsto J$ if I is an ID and J is an ID that follows from I after one move of the TM
- Technically: $\mapsto$ is a relation on IDs, defined by the δ function for the TM
- Then for any $x \in \Gamma^*$, $c \in \Gamma$, $q \in Q$, $a \in \Gamma$, and $y \in \Gamma^*$, we have two kinds of moves:
  - Left moves: if $\delta(q,a) = (p,b,L)$ then $xcqay \mapsto idfix(xpcby)$
  - Right moves: if $\delta(q,a) = (p,b,R)$ then $xcqay \mapsto idfix(xcbpy)$

Zero-Or-More-Move Relation

- As we did with grammars, NFAs, and stack machines, we extend this to a zero-or-more-move $\mapsto^*$
- Technically, $\mapsto^*$ is a relation on IDs, with $I \mapsto^* J$ if and only if there is a sequence of zero or more relations that starts with I and ends with J
- Note this is reflexive by definition: we always have $I \mapsto^* I$ by a sequence of zero moves

The Language Accepted By A TM

- *idfix*(q_0x) is the initial ID of *M*, given input $x \in \Sigma^*$
- (*idfix*(q_0x) = **B**q_0x if $x \neq \varepsilon$, or **B**q_0**B** if x = ε)
- Then *x* is accepted if and only if *M* has a sequence of zero or moves from *idfix*(q_0x) to some ID in an accepting state
  - Regardless of what is left on the tape
  - Regardless of the final position of the head
- Technically,
  $L(M) = \{x \in \Sigma^* \mid idfix(q_0x) \mapsto^* ypz \text{ for some } p \in F\}$

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Three Possible Computation Outcomes for a TM

- It halts and accepts:
  - It halts and accepts if it reaches an accepting state; even if there are transitions leaving the accepting state, since they have no effect on $L(M)$
- It halts and rejects (it gets stuck):
  - It halts and rejects if does not reach an accepting state, but gets into an ID I with no J for which $I \mapsto J$
- And there is a third possibility: it runs forever:
  - It always has a move, but never reaches an accepting state

Example:

- $M = (\{q, r, s\}, \{0,1\}, \{0,1,\mathbf{B}\}, \delta, \mathbf{B}, q, \{s\})$:
  - $\delta(q,1) = (r,1,R)$
  - $\delta(q,0) = (s,0,R)$
  - $\delta(r,\mathbf{B}) = (q,\mathbf{B},L)$
- Given input 0, *M* halts and accepts:
  - $\mathbf{B}q0 \mapsto 0s\mathbf{B}$
  - In fact, $L(M) = L(0(0+1)^*)$ [accepts any string of 0s and 1s if it starts with a 0]
- Given input ε, *M* halts and rejects:
  - $\mathbf{B}q\mathbf{B} \mapsto$?
- Given input 1, *M* runs forever:
  - $\mathbf{B}q1 \mapsto 1r\mathbf{B} \mapsto \mathbf{B}q1 \mapsto 1r\mathbf{B} \mapsto \mathbf{B}q1 \mapsto 1r\mathbf{B} \mapsto \ldots$

Running Forever

- We can make a TM for $L(0(0+1)^*)$ that halts on all inputs
- In general, though, it is not always possible for TMs to avoid infinite loops, as we will prove in later chapters
- The risk of running forever is the price TMs pay for their great power

Earlier Infinite Loops

- NFAs could in some sense run forever, since they can contain cycles of ε-transitions
- Same with stack machines
- But these were always avoidable:
  - Any NFA could be converted into one without cycles of ε-transitions (in fact, into a DFA, without ε-transitions at all)
  - Similarly, one can show that for any CFL there is a stack machine without cycles of ε-transitions
- For TMs they are not avoidable…

Three-Way Partition

- The three possible TM outcomes partition Σ^* into three subsets
- So instead of just defining $L(M)$, a TM really defines three languages:
  - The language accepted by a TM:
    $L(M) = \{x \in \Sigma^* \mid idfix(q_0x) \mapsto^* ypz \text{ for some } p \in F\}$
  - The language rejected by a TM:
    $R(M) = \{x \in \Sigma^* \mid x \notin L(M) \text{ and there is some ID } I \text{ with } idfix(q_0x) \mapsto^* I \text{ and no } J \text{ with } I \mapsto J\}$
  - The language on which a TM runs forever:
    $F(M) = \{x \in \Sigma^* \mid x \notin L(M) \text{ and } x \notin R(M)\}$

Recursive and RE

- A TM *M* is a *total TM* if and only if *F*(*M*) = {}
  - A total TM never loops forever
- A *recursive* language is one that is *L*(*M*) for some *total* TM M (it never loops forever)
- A *recursively enumerable* (RE) language is one that is *L*(*M*) for some TM M (it can loop forever)
- We will see that these two sets of languages are not the same; some languages are RE but not recursive
- The names are odd, but standard:
  - RE and recursive languages were identified in mathematical studies of computability using recursive function theory

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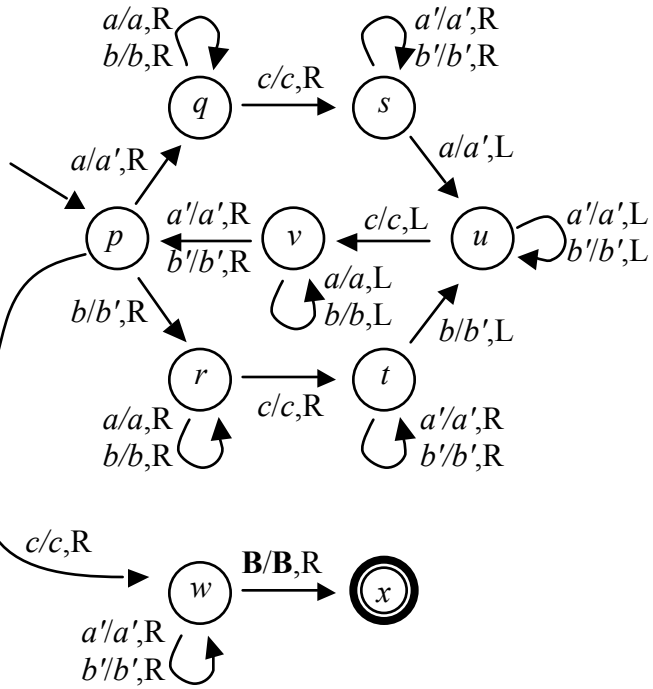
$\{xcx \mid x \in \{a,b\}^*\}$

- Not context-free, as we've seen
- A TM for this language can work by checking each symbol in the first *x* against the corresponding symbol in the second *x*
- To keep track of where it is, it will *mark* those symbols that have already been checked
- That is, it will overwrite them with marked versions of the same symbols, for instance by overwriting *a* with *a′*

A TM For $\{xcx \mid x \in \{a,b\}^*\}$



Example

- See example slides

Two General Techniques

- Marking:
  - A cell can be marked by overwriting the symbol with a marked version of that symbol
  - Our input alphabet was {*a*,*b*,*c*}, but the tape alphabet was {*a*,*b*,*c*,*a′*,*b′*,**B**}
- Remembering
  - A TM can use states to record any finite information
  - Ours remembered whether *a* or *b* was seen in the first half, using two paths of states

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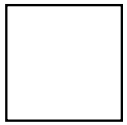
A 3-Tape TM

- Simulations are easier if we have a 3-tape TM
- Like the basic model, but with three tapes, each with an independent read/write head
- Basic model: $\delta(q,X) = (p,Y,D)$
  - Two inputs: current state and symbol at head
  - Three outputs: new state, symbol to write, and direction
- 3-tape model: $\delta(q,X_1,X_2,X_3) = (p,Y_1,D_1,Y_2,D_2,Y_2,D_3)$
  - Separate read, write, and direction for each of the three heads
- Otherwise, the same

Same Power

- The 3-tape model is easier to program, but no more powerful than the basic model
- For any 3-tape TM we can construct an equivalent basic TM
- We can encode all the information from the 3 tapes (with their head positions) in a single tape, using an enlarged alphabet…



- Image the three tapes side by side, each with its current head position
- Encode all this information on one tape, using triples as symbols in the enlarged tape alphabet:

1-Tape TM Construction

- Given any 3-tape M_3, construct a 1-tape M_1
  - Use the alphabet of triples, with (**B**,**B**,**B**) as blank
  - To simulate a move of M_3, M_1 makes two passes:
    - A left-to-right pass, collecting the three input symbols. Keep track (using state) of M_3's state and input symbols. Now if M_3 accepts, halt and accept; if M_3 rejects, halt and reject.
    - A right-to-left pass, carrying out M_3's actions at each of its three head positions (writing and moving marks). Leave the head at the leftmost mark and go to step 1.
- Far more states, symbols, and moves than M_3
- But for any input string, M_1's outcome matches M_3's

Theorem 16.8

For any given partition of a Σ^* into three subsets L, R, and F, there is a three-tape TM M_3 with $L(M_3) = L$, $R(M_3) = R$, and $F(M_3) = F$, if and only if there is a one-tape TM M_1 with $L(M_1) = L$, $R(M_1) = R$, and $F(M_1) = F$.

- Proof sketch:
  - Given any 3-tape TM we can construct a 1-tape TM that simulates it, as just outlined
  - Given any 1-tape TM we can construct a 3-tape TM that simulates it, simply by not using two of its tapes

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Automata As Input

- Our goal is to make TMs that can simulate other automata, given as input
- TMs can only take strings as input, so we need a way to encode automata as strings
- We'll start with the simplest: DFAs…

DFAs Encoded Using {0,1}

- The DFA's alphabet and strings:
  - Number Σ arbitrarily as $\Sigma = \{\sigma_1, \sigma_2, ...\}$
  - Use the string 1^i to represent symbol σ_i
  - Use 0 as a separator for strings
  - For example, if $\Sigma = \{a, b\}$, let a = σ_1 and b = σ_2; then *abba* is represented by 101101101
- The DFA's states:
  - Number $Q = \{q_1, q_2, ...\}$, making q_1 the start state and numbering the others arbitrarily
  - Use the string 1^i to represent symbol q_i

DFA Encoding, Continued

- The DFA's transition function:
  - Encode each transition $\delta(q_i,\sigma_j) = q_k$ as a string $1^i01^j01^k$
  - Encode the entire transition function as a list of such transitions, in any order, using 0 as a separator
  - For example,

  - Numbering *a* as σ_1 and *b* as σ_2, δ is
    $\delta(q_1,\sigma_1) = q_2$ $\quad$ $\delta(q_1,\sigma_2) = q_1$
    $\delta(q_2,\sigma_1) = q_1$ $\quad$ $\delta(q_2,\sigma_2) = q_2$
  - That is encoded as:
    101011 0 101101 0 110101 0 11011011

DFA Encoding, Continued

- The DFA's set of accepting states:
  - We already encode each state q_i as 1^i
  - Use a list of state codes, separated by 0s
- Finally, the complete DFA:
  - Transition-function string, 00, accepting-state string:
    101011 0 101101 0 110101 0 11011011 00 11

Simulating a DFA

- We have a way to represent a DFA as a string over {0,1}
- Now, we'll show how to construct a TM that simulates any given DFA
  - Given the encoded DFA as input, along with an encoded input string for it
  - Decide whether the given DFA accepts the given string
- We'll use a 3-tape TM…

3-Tape DFA Simulator

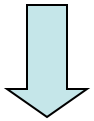
- First tape holds the DFA being simulated
- Second tape holds the DFA's input string
- Third tape hold the DFA's current state q_i, encoded as 1^i as usual

Example:

- Initial configuration, in the start state, on input *abab*:

△

- Each simulated move performs one state transition and erases one encoded input symbol…



First move on *abab*: read *a*, go to state q_2

Strategy

- Step 1: handle termination:
  - If the second tape is not empty, go to step 2
  - If it is empty, the DFA is done; search the list of accepting states (tape 1) for a match with the final state (tape 3)
  - If found, halt and accept; if not, halt and reject
- Step 2: look up move:
  - Search tape 1 for the move $1^i01^j01^k$ that applies now, where 1^i matches the current state (tape 3) and 1^j matches the current input symbol (tape 2)
- Step 3: execute move:
  - Replace the 1^i on the tape 3 with 1^k
  - Write **B** over the 1^j (and any subsequent 0) on tape 2
  - Go to step 1

An Easy Simulation

- That was no challenge for our 3-tape TM
- Used only a fixed, finite portion of each tape
- There is (by Theorem 16.8) a 1-tape TM with the same behavior
- One detail we're skipping: what should the TM do with ill-formed inputs?
  - If we specified behavior for ill-formed inputs, there would have to be an extra initial pass to verify the proper encoding of a DFA and its input

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Simulating Other Automata

- We can use the same 3-tape technique to simulate all our other automata
- Trickier for nondeterministic models (NFAs and stack machines): our deterministic TM must search all sequences of moves
- Relatively straightforward for deterministic automata

Language Categories

- Proofs of these inclusions can be constructed by having total TMs simulate other automata

Universal Turing Machines

- A *universal Turing machine* is any Turing machine that takes an encoded Turing machine and an encoded input string, and decides whether the given Turing machine accepts the given string
- It's like an interpreter: a program that takes another program as input and carries out the instructions of that input program
- A universal TM is, in effect, a TM interpreter

A Universal TM Outline

- Design a TM encoding using {0,1}:
  - Most of it is the transition function, like our DFA encoding
- Familiar 3-tape layout:
  - Tape 1: the encoded TM as input
  - Tape 2: that TM's tape (input and working space)
  - Tape 3: that TM's current state
- Simulation:
  - Look up the appropriate transition (on tape 1)
  - Do the necessary write and move (on tape 2)
  - Do the state change (on tape 3)
  - Repeat until accepting state or no next move

Constructions?

- All the “constructions” since 16.7 have been high-level outlines, not detailed constructions:
  - 3-tape to 1-tape conversion
  - DFA simulator
  - Universal TM
- In effect, we sketched proofs that the constructions were possible, without actually doing them
- Why not give more detail?

The Problem Of Detail

- It is hard to figure out what a TM does by inspection
  - Very hard for small TMs
  - Inhumanly difficult for large TMs
- To convince someone that a language can be recognized by a TM, it is not often useful to just show the TM that does it
- It is more convincing to give less detail -- to describe in outline how a TM might work
- Once you're convinced a TM can be constructed for a given problem, there is no point in actually constructing it.