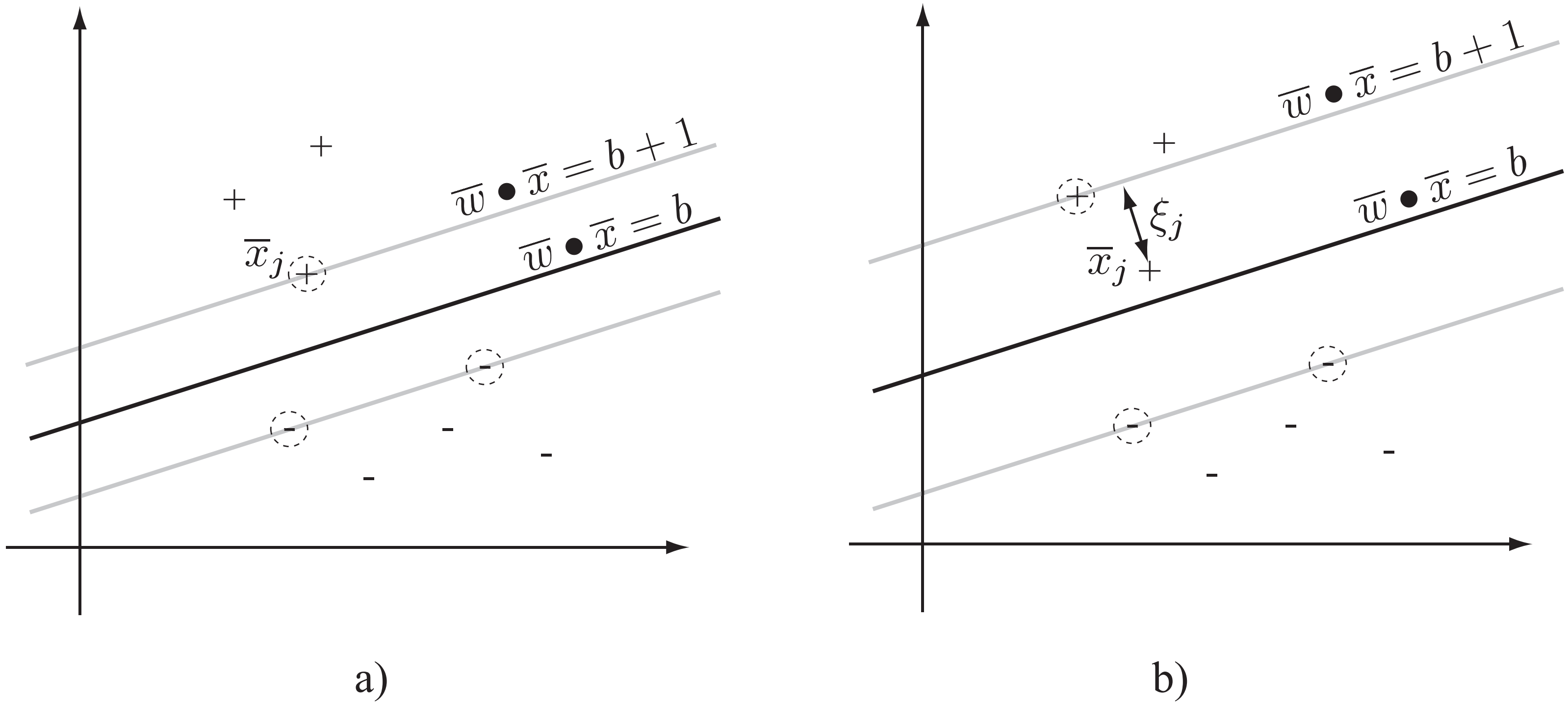
Noisy Data

Noisy data $\Rightarrow$ small margin.



Solution: ignore the noisy points.

Maximum Margin Classifiers

Recall that our maximum margin classifiers are models of the form

$$\hat{f}(\overline{x}) = \operatorname{sign}\left(\overline{w} \bullet \overline{x} - b\right),$$

where the normal vector $\overline{w}$ and the offset term b of the decision surface are computed via the primal optimization problem,

$$\min \phi(\overline{w}, b) = \min \frac{1}{2}\overline{w} \bullet \overline{w},$$

subject to the constraints,

$$y_i(\overline{w} \bullet \overline{x}_i - b) - 1 \geq 0,$$

with $i = 1, \ldots, l$ given the training set $(\overline{x}_1, y_1), \ldots, (\overline{x}_l, y_l) \in \mathbb{R}^n \times \{+1, -1\}$.

Softmargin Classifiers

If we allow points to lie on the “wrong” side of their supporting hyperplanes we need to keep track of the amount of error that this introduces $\Rightarrow$ *slack variables* denoted with ξ (xi) (see Fig b above)

We change our training algorithm by taking the slack variables into account. We rewrite our constraints as

$$y_i(\overline{w} \bullet \overline{x}_i - b) + \xi_i - 1 \geq 0,$$

with $\xi_i \geq 0$.

We also modify our objective function,

$$\min_{\overline{w},\overline{\xi},b} \phi(\overline{w}, \overline{\xi}, b) = \min_{\overline{w},\overline{\xi},b} \left(\frac{1}{2} \overline{w} \bullet \overline{w} + C \sum_{i=1}^{l} \xi_i \right),$$

Our new objective function looks just like the objective function for maximum margin classifiers except for the penalty term $C \sum_{i=1}^{l} \xi_i$. C is called the *cost*. In this way the optimization becomes a trade off between the size of the margin and the size of the error measured by the slack variables,

large C $\sim$ small margin	
small C $\sim$ large margin	

Softmargin Classifiers

Putting this all together,

Proposition: [Soft-Margin Optimization] Given a training set

$$D = \{(\overline{x}_1, y_1), (\overline{x}_2, y_2), \ldots, (\overline{x}_l, y_l)\} \subseteq \mathbb{R}^n \times \{+1, -1\},$$

we can compute a soft-margin decision surface $\overline{w}^* \bullet \overline{x} = b^*$ with an optimization,

$$\min_{\overline{w}, \overline{\xi}, b} \phi(\overline{w}, \overline{\xi}, b) = \min_{\overline{w}, \overline{\xi}, b} \left(\frac{1}{2} \overline{w} \bullet \overline{w} + C \sum_{i=1}^{l} \xi_i \right),$$

subject to the constraints,

$$\begin{aligned} y_i(\overline{w} \bullet \overline{x}_i - b) + \xi_i - 1 &\geq 0, \\ \xi_i &\geq 0, \end{aligned}$$

with $i = 1, \ldots, l$, $\overline{\xi} = (\xi_1, \ldots, \xi_l)$, and $C > 0$.

Note: The slack variables have no impact on the form of our model $\hat{f}(\overline{x}) = \text{sign}(\overline{w}^* \bullet \overline{x} - b^*)$.

The Dual

As before we start by constructing the Lagrangian,

$$L(\overline{\alpha}, \overline{\beta}, \overline{w}, \overline{\xi}, b) = \frac{1}{2}\overline{w} \bullet \overline{w} + C\sum_{i=1}^{l} \xi_i$$

$$- \sum_{i=1}^{l} \alpha_i (y_i(\overline{w} \bullet \overline{x}_i - b) + \xi_i - 1)$$

$$- \sum_{i=1}^{l} \beta_i \xi_i$$

We have an additional set of Lagrangian multipliers for the additional constraints.

This gives us the Lagrangian optimization problem,

$$\max_{\overline{\alpha}, \overline{\beta}} \min_{\overline{w}, \overline{\xi}, b} L(\overline{\alpha}, \overline{\beta}, \overline{w}, \overline{\xi}, b),$$

subject to the constraints,

$$\alpha_i \geq 0,$$

$$\beta_i \geq 0,$$

for $i = 1, \ldots, l$.

The Dual

Since the primal objective function is convex, this Lagrangian has a unique saddle point and therefore a solution $\overline{\alpha}^*, \overline{\beta}^*, \overline{w}^*, \overline{\xi}^*, b^*$ has to satisfy the KKT conditions,

$$\begin{aligned}
\frac{\partial L}{\partial \overline{w}}(\overline{\alpha}, \overline{\beta}, \overline{w}^*, \overline{\xi}, b) &= 0, \\
\frac{\partial L}{\partial \xi_i}(\overline{\alpha}, \overline{\beta}, \overline{w}, \xi_i^*, b) &= 0, \\
\frac{\partial L}{\partial b}(\overline{\alpha}, \overline{\beta}, \overline{w}, \overline{\xi}, b^*) &= 0, \\
\alpha_i^*(y_i(\overline{w}^* \bullet \overline{x}_i - b^*) + \xi_i^* - 1) &= 0, \\
\beta_i^* \xi_i^* &= 0, \\
y_i(\overline{w}^* \bullet \overline{x}_i - b^*) + \xi_i^* - 1 &\geq 0, \\
\alpha_i^* &\geq 0, \\
\beta_i^* &\geq 0, \\
\xi_i^* &\geq 0,
\end{aligned}$$

for $i = 1, \ldots, l$.

The Dual

Now taking the partial derivatives in terms of the primal variables:

$$\frac{\partial L}{\partial \overline{w}}(\overline{\alpha}, \overline{\beta}, \overline{w}^*, \overline{\xi}, b) = \overline{w}^* - \sum_{i=1}^{l} \alpha_i y_i \overline{x}_i = \overline{0},$$

$$\frac{\partial L}{\partial b}(\overline{\alpha}, \overline{\beta}, \overline{w}, \overline{\xi}, b^*) = \sum_{i=1}^{l} \alpha_i y_i = 0,$$

$$\frac{\partial L}{\partial \xi_i}(\overline{\alpha}, \overline{\beta}, \overline{w}, \xi_i^*, b) = C - \alpha_i - \beta_i = 0,$$

Since both $\alpha_i \geq 0$ and $\beta_i \geq 0$ the last equation implies that

$$C \geq \alpha_i \geq 0.$$

Putting this all together we can derive the dual,

$$\phi'(\overline{\alpha}) = \sum_{i=1}^{l} \alpha_i - \frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{l} \alpha_i \alpha_j y_i y_j \overline{x}_i \bullet \overline{x}_j.$$

The Dual

Proposition [The Soft-Margin Lagrangian Dual] Given a soft-margin optimization in primal form (see the beginning of this set of slides) then the Lagrangian dual optimization for a soft-margin classifier is

$$\max_{\overline{\alpha}} \phi'(\overline{\alpha}) = \max_{\overline{\alpha}} \left(\sum_{i=1}^{l} \alpha_i - \frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{l} \alpha_i \alpha_j y_i y_j \overline{x}_i \bullet \overline{x}_j \right)$$

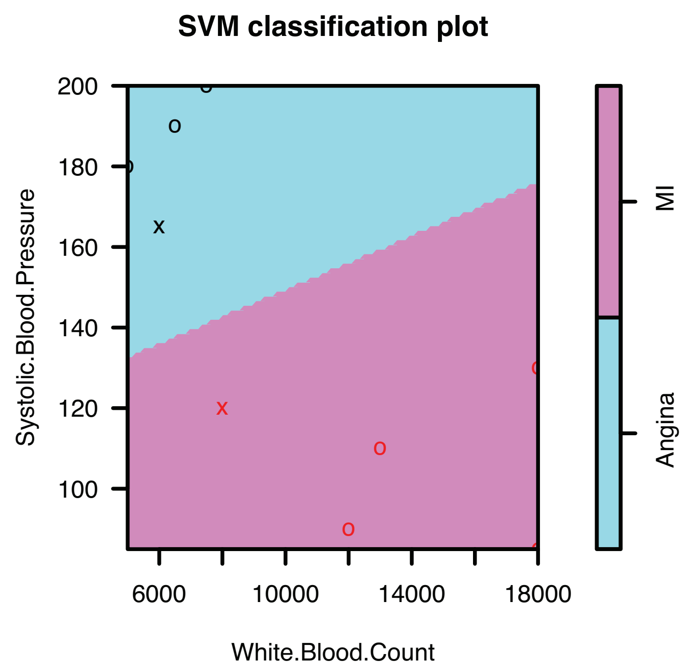
subject to the constraints,

$$\sum_{i=1}^{l} \alpha_i y_i = 0,$$
$$C \geq \alpha_i \geq 0,$$

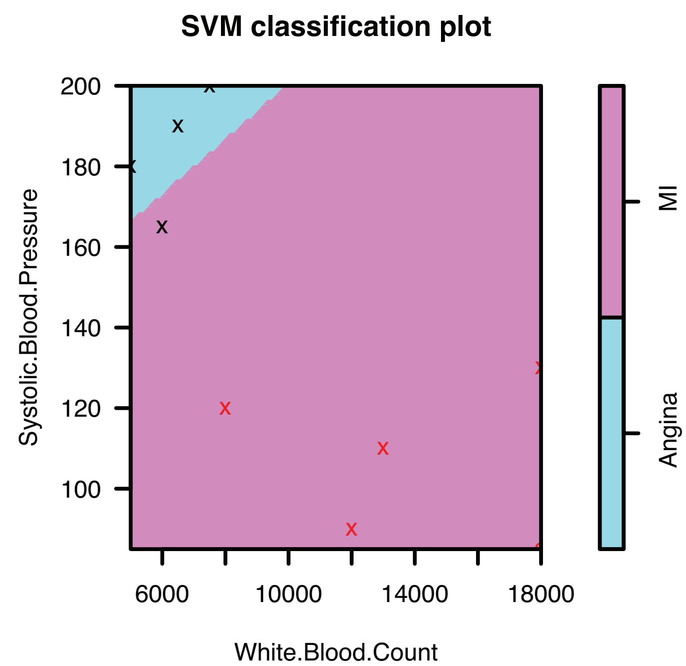
with $i = 1, \ldots, l$. Here, C is the cost constant.

It is remarkable that this dual differs from the hard-margin case only in the range of values the Lagrangian multipliers can take on: Points in the margin $\alpha_i = C$, points on the supporting hyperplanes $C > \alpha_i > 0$, and points far away from the decision surface $\alpha_i = 0$.

Soft-Margin Classifiers



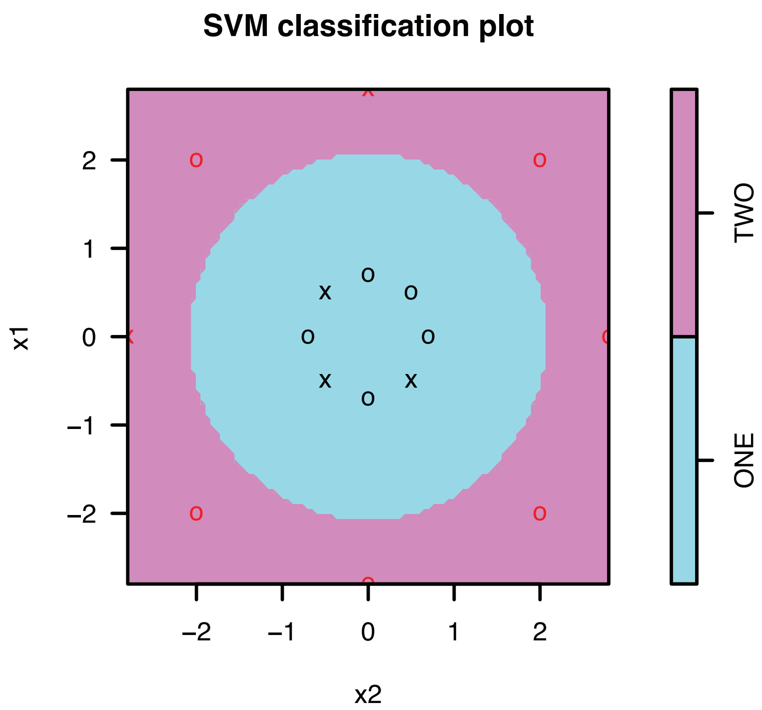
a)



b)

```
> svm.model <- svm(Diagnosis˜.,
                  data=biomed.df,
                  type="C-classification",
                  cost=1.0,
                  kernel="linear")
```

Soft-Margin Classifiers




```
> svm.model <- svm(y˜.,
                   data=non.linear.df,
                   type="C-classification",
                   cost=1,
                   kernel="polynomial",
                   degree=2,
                   coef0=0)
```

Kernel-Perceptron

let $D = \{(\overline{x}_1, y_1), \ldots, (\overline{x}_l, y_l)\}$
let $0 < \eta < 1$
$\overline{\alpha} \leftarrow \overline{0}$
$b \leftarrow 0$
$r \leftarrow \max\{|\overline{x}| \mid (\overline{x}, y) \in D\}$
repeat
  for $i = 1$ **to** l
    if $\text{sign}(\sum_{j=1}^{l} \alpha_j y_j \overline{x}_j \bullet \overline{x}_i - b) \neq y_i$ **then**
      $\alpha_i \leftarrow \alpha_i + 1$
      $b \leftarrow b - \eta y_i r^2$
    end if
  end for
until done
return $(\overline{\alpha}, b)$

let $D = \{(\overline{x}_1, y_1), \ldots, (\overline{x}_l, y_l)\}$
let $\eta > 0$
$\overline{\alpha} \leftarrow \overline{0}$
$b \leftarrow 0$
repeat
  for $i = 1$ **to** l **do**
    if $\text{sign}(\sum_{j=1}^{l} \alpha_j y_j k(\overline{x}_j, \overline{x}_i) - b) \neq y_i$ **then**
      $\alpha_i \leftarrow \alpha_i + 1$
      $b \leftarrow b - \eta y_i$
    end if
  end for
until done
return $(\overline{\alpha}, b)$

Observations:

- We extend our linear classifier to a non-linear perceptron.
- However, sub-optimal decision surface, algorithm stops as soon as a decision surface is found.