The Mathematics of Functions

The λ-calculus is very low level. Here we investigate functions in a more abstract (mathematical) setting.

This is very similar to writing algorithms in English prose rather than writing actual machine code for the Turing machines.

The Mathematics of Functions

Function Application and Composition:

Let $f: \ A \longrightarrow B$ be a (total) function from A to B, then for every value $x \in A$ we obtain a value $y \in B$,

$$fx = y$$

Function application is expressed by the *juxtaposition* of the function and its argument and is evaluated from right to left.

Now assume that we have another function $g: \ B \longrightarrow C$ from B into C, then we can apply the function g to the result of f. For every value in $x \in A$ we obtain a value $z \in C$,

$$gfx = gy = z$$

In other words, we just constructed a new function, call it $h: \ A \longrightarrow C$, such that

$$hx = gfx = gy = z$$

We can express the same idea using function composition, $\circ$, without having to explicitly reference any values in A, B, or C,

$$h = g \circ f$$

and we say that "h is the composition of the function g with function f".

Note, that function composition is computed from right to left.

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The Tuple:

Given two elements $x \in A$ and $y \in B$, then the tuple constructs an element of the cross-product $A \times B$,

$$x \in A, y \in B \Rightarrow (x, y) \in A \times B$$

This is an important construction because it lets us apply functions to pairs (tuples) of values. Assume we have a function $f: A \times B \to C$. This function can only be applied to values in the cross-product $A \times B$, but we know how to construct these values – yes, the tuple,

$$f(x, y) = z$$

for $x \in A$, $y \in B$, and $z \in C$.

Notice that we consider the tuple and function application separate computational steps.

We can of course generalize this to arbitrarily complex tuples, let $x_1 \in X_1, \ldots, x_n \in X_n$ and let $f: X_1 \times \ldots \times X_n \to Y$, then

$$f(x_1, \ldots, x_n) = y$$

for $y \in Y$.

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Tuples of Functions:

Something interesting happens when we construct tuples of functions, let $f : \; A \longrightarrow B$ and let $g : \; A \longrightarrow B$, then

$$(f, g) \in (A \longrightarrow B) \times (A \longrightarrow B)$$

The pair of functions acts in parallel on an input in A and produces a pair of output values in $B \times B$.

Let $a \in A$ and $b_f, b_g \in B$, then

$$(f, g)a = (fa, ga) = (b_f, b_g)$$

with $fa = b_f$ and $ga = b_g$.

Something a little bit more complicated. Let $f : \; X \times Y \longrightarrow Z$ and $g : \; X \times Y \longrightarrow Z$ with $x \in X, y \in Y$ and $z_1, z_2 \in Z$,

$$(f, g)(x, y) = (f(x, y), g(x, y)) = (z_1, z_2)$$

where $f(x, y) = z_1$ and $g(x, y) = z_2$.

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Projection Functions:

Given a tuple

$$(x_1, \ldots, x_n)$$

with $x_1 \in X_1, \ldots, x_n \in X_n$, then we can project the i^{th} component of the tuple with the projection function $p_i^{(n)} : X_1 \times \ldots \times X_n \rightarrow X_i$,

$$p_i^{(n)}(x_1, \ldots, x_n) = x_i$$

with $1 \leq i \leq n$.

A more concrete example, let $x \in A$ and $y \in B$, then

$$p_1^{(2)}(x, y) = x$$

$$p_2^{(2)}(x, y) = y$$

but

$$p_1^{(3)}(x, y) = ???$$

$$p_5^{(2)}(x, y) = ???$$

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Putting Projection Functions, Tuples, and Composition together:

Let $f: X_1 \times X_2 \longrightarrow Y$ and let $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$, then

$$f \circ (p_1^{(3)}, p_2^{(3)}) : X_1 \times X_2 \times X_3 \longrightarrow Y$$

Applying this function to our tuple we have

$$f(p_1^{(3)}, p_2^{(3)})(x_1, x_2, x_3) = f(p_1^{(3)}(x_1, x_2, x_3), p_2^{(3)}(x_1, x_2, x_3)) = f(x_1, x_2)$$

Here is another example, let $(x_1, x_2) \in X_1 \times X_2$, then

$$(p_1^{(2)}, p_2^{(2)})(x_1, x_2) = (p_1^{(2)}(x_1, x_2), p_2^{(2)}(x_1, x_2)) = (x_1, x_2)$$