Cook-Levin Theorem

In the 1970's Stephen Cook and Leonid Levin independently discovered that there are problems in NP whose complexity are related to all other problems in NP – these problems are called NP-complete problems.

As we have seen, NP-complete problems are related to other NP problems via polynomial reductions.

The first and most famous NP-complete problem discovered was a problem around the satisfiability of logic formulas.

SAT

A Boolean formula is an expression involving Boolean variables (x, y, *etc.*) and operations ($\wedge$, $\vee$, $\neg$, where $\neg x = \overline{x}$),

$$\phi = (\overline{x} \wedge y) \vee (x \wedge \overline{z}).$$

A Boolean formula is satisfiable if some assignment of $true$ and $false$ to the variables of the formula makes the formula evaluate to $true$.

For example, the assignment

$$x = false$$
$$y = true$$
$$z = false$$

will make ϕ above evaluate to true.

The ***satisfiability problem*** is to test whether a Boolean formula is satisfiable, that is

$$SAT = \{\langle\phi\rangle | \phi \text{ is a satisfiable Boolean formula}\}.$$

SAT

Theorem: (Cook-Levin)

$$SAT \in NP\text{-complete.}$$

Proof Sketch: For an NP-complete problem we need to show that it is in NP and that all $A \in NP$ reduce to it.

(a) It is easy to see that a truth assignment to the variables of a formula can be checked in polynomial time.

(b) We need to show that $A \leq_p SAT$ for all $A \in NP$. This is done by simulating the computations of a NTM deciding A on some string w using Boolean formulas such that

$$w \in A \text{ iff } f(w) \in SAT$$

where f converts the string w into the Boolean formula $f(w)$.[a] □

Note: In some sense this reinforces our notion that first-order logic is a powerful language to reason about complex problems.

[a] For details, please see the Cook-Levin Theorem in the book.

$3SAT$

The $3SAT$ problem:

- A special case of SAT,
- Formulas are in *conjunctive normal form* (cnf),

$$\underbrace{(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4)}_{\text{clause}} \overbrace{\wedge}^{\text{conjunction}} (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6}),$$

- 3cnf – each clause in the cnf has only 3 literals (or variables),

$$(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_3 \vee \overline{x_5} \vee x_6),$$

We define,

$$3SAT = \{\langle\phi\rangle | \phi \text{ is a satisfiable 3cnf formula}\}.$$

$3SAT$

Theorem:

$$3SAT \in NP\text{-complete.}$$

Proof Sketch: We show this by constructing a polynomial time reduction from SAT to $3SAT$.

First, observe that any formula $\phi \in SAT$ can be rewritten in cnf such that $\hat{\phi} = c_1 \wedge c_2 \wedge \ldots \wedge c_m$ where each clause c_i is a disjunction of Boolean variables, say $z_1 \ldots z_n$.

We now construct the polynomial time reduction $f : SAT \to 3SAT$ such $f(\hat{\phi}) = \phi_{3SAT}$. We replace each c_i in $\hat{\phi}$ by a collection of literal clauses over the variables which appear in c_i plus some additional variables which appear only in these 3 literal clauses. More specifically, let $c_i = z_1 \vee z_2 \vee \ldots \vee z_k$ where each z_j is a Boolean variable, then

$k = 1$: Here $c_i = z_1$. Use the additional variables $y_{i,1}$ and $y_{i,2}$ to construct the clauses in 3cnf

$$(z_1 \vee y_{i,1} \vee y_{i,2}) \wedge (z_1 \vee \overline{y_{i,1}} \vee y_{i,2}) \wedge (z_1 \vee y_{i,1} \vee \overline{y_{i,2}}) \wedge (z_1 \vee \overline{y_{i,1}} \vee \overline{y_{i,2}})$$

$k = 2$: Here $c_i = (z_1 \vee z_2)$. Use the additional variable $y_{i,1}$ to construct the clauses in 3cnf

$$(z_1 \vee z_2 \vee y_{i,1}) \wedge (z_1 \vee z_2 \vee \overline{y_{i,1}})$$

$k = 3$: Here $c_i = (z_1 \vee z_2 \vee z_3)$, already in 3cnf, nothing to do.

$3SAT$

$k = 4$: Here $c_i = (z_1 \vee z_2 \vee z_3 \vee z_4)$, use the additional variable $y_{i,1}$ to construct the clauses in 3cnf

$$(z_1 \vee z_2 \vee y_{i,1}) \wedge (\overline{y_{i,1}} \vee z_3 \vee z_4)$$

$k > 4$: Here $c_i = (z_1 \vee z_2 \vee \ldots \vee z_k)$, use the additional variables $y_{i,1}, y_{i,2}, \ldots, y_{i,k-3}$ to construct the clauses in 3cnf

$$\begin{array}{rc} (z_1 \vee z_2 \vee y_{i,1}) & \wedge \\ (\overline{y_{i,1}} \vee z_3 \vee y_{i,2}) & \wedge \\ (\overline{y_{i,2}} \vee z_4 \vee y_{i,3}) & \wedge \\ (\overline{y_{i,3}} \vee z_5 \vee y_{i,4}) & \wedge \\ \ldots & \wedge \\ (\overline{y_{i,k-3}} \vee z_{k-1} \vee z_k) & \end{array}$$

We now show that f is a polynomial time reduction. First observe that the **maximum** number of variables that can occur in a clause of $\hat{\phi}$ is n. Also observe that there are m clauses in $\hat{\phi}$. Therefore, the maximum number of conversions is bounded by $O(nm)$ which is clearly polynomial. Considering that the above transformation can also be accomplished in polynomial time we conclude that f is a polynomial time function.

$3SAT$

We now show that

$$\hat{\phi} \in SAT \text{ iff } f(\hat{\phi}) \in 3SAT$$

For the case that $k \leq 4$ we note that whenever $\hat{\phi}$ is satisfied then so is $f(\hat{\phi})$. For the reverse we note that if $f(\hat{\phi})$ is satisfied we simply restrict the assignments to the variables that appear in $\hat{\phi}$ to obtain an assignment that satisfies $\hat{\phi}$.

For the case that $k > 4$, given a truth assignment in some clause c_i in $\hat{\phi}$, the case of $k = 4$ is easily generalized to this case.

Thus, the satisfiability of $\hat{\phi}$ implies the satisfiability of $f(\hat{\phi})$. To see the converse, simply restrict a satisfying assignment to variables in $f(\hat{\phi})$ to the variables occurring in $\hat{\phi}$. $\square$

$CLIQUE$

Theorem:

$$CLIQUE \in NP\text{-complete}$$

where $CLIQUE = \{\langle G, k\rangle | G$ is an undirected graph with a k-clique $\}$.

Proof: We prove this by a polynomial reduction f from $3SAT$ to $CLIQUE$, such that

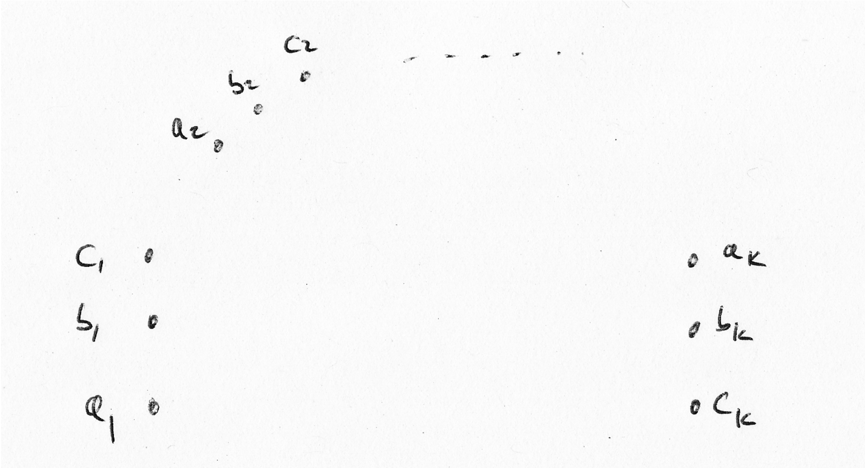
$$\phi_k \in 3SAT \text{ iff } f(\phi_k) \in CLIQUE,$$

where ϕ_k is a 3cnf formula with k clauses and $f(\phi_k) = \langle G, k\rangle$.

Given

$$\phi_k = (a_1 \vee b_1 \vee c_1) \wedge (a_1 \vee b_1 \vee c_1) \wedge \ldots (a_k \vee b_k \vee c_k).$$

we construct the nodes of the graph as



CLIQUE

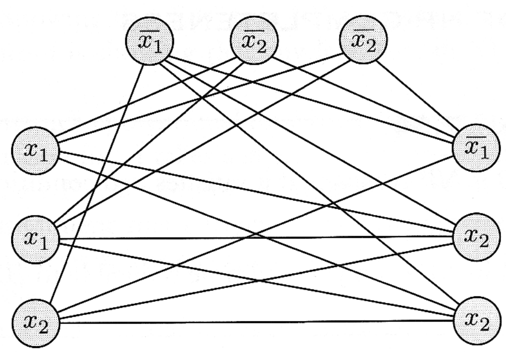
We construct the edges by connecting all the nodes except for

- nodes that are in the same triple, and
- nodes that have contradictory labels, i.e., x and $\overline{x}$.

Example construction using

$$\phi_3 = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$$

gives rise to the graph



$CLIQUE$

It is easy to see that this is a polynomial time construction (let n be the number of nodes then we see that algorithm runs in $O(n^2)$ time).

We now have to verify the reduction condition

$$\phi_k \in 3SAT \text{ iff } f(\phi_k) \in CLIQUE$$

First the '⇒' direction, suppose that ϕ_k has a satisfying assignment, that means that each clause has at least one literal that is true. In each triple of G we choose one node that corresponds to a true literal. The number of nodes selected is k, one in each triple. All the selected nodes are connected by edges. This shows that a satisfying assignment of ϕ_k produces a k-clique.

For the converse, assume that G has a k-clique. By construction, no two nodes can occur in the same triple. Therefore, each of the k triple contain exactly one of the k clique nodes. Each node in the k-clique denotes an assignment to true for a literal in ϕ_k. This is always true because opposing literals are not connected. $\square$

$HAMPATH$

Recall that $HAMPATH = \{\langle G, s, t\rangle | G$ is a directed graph with a Hamiltonian path from s to $t\}$ and a Hamiltonian path is a path in a graph that goes through each node exactly once.

Theorem:

$$HAMPATH \in NP\text{-complete.}$$

Proof Sketch: We know that $HAMPATH \in NP$. It remains to show that $A \leq_p HAMPATH$, for all $A \in NP$. We show this by a polynomial time reduction f from $3SAT$ to $HAMPATH$,

$$\phi_k \in 3SAT \text{ iff } f(\phi_k) \in HAMPATH,$$

where $f(\phi_k) = \langle G, s, t\rangle$.

For details of the construction see the book. . .

NP-Hard

Definition: A language Q is ***NP-hard*** if it satisfies two conditions:

1. $Q \notin NP$, and
2. every $Q_i \in NP$ is polynomial time reducible to Q.

Traveling Salesman

The idea is, given a set of cities (nodes) that are connected via roads (weighted edges), find the cheapest route through all the cities (find a Hamiltonian path that minimizes the sum of the weights in the path). Formally,

$TSP = \{\langle G, s, t, w \rangle | G$ is directed weighted graph with a *minimal* Hamiltonian path of weight w from s to $t\}$.

Traveling Salesman

Theorem:

$$TSP \in NP\text{-hard.}$$

Proof: Note, a problem is NP-hard if every $L \in NP$ can be reduced to it in polynomial time but the problem itself is not in NP. No known NP solution exists for TSP (NP problems have polynomial time verifiers; in TSP it is not possible to verify a certificate in polynomial time). It remains to show that all $L \in NP$ reduce to it in polynomial time. We will show this by a polynomial time reduction f from $HAMPATH$ to TSP,

$$\langle G, s, t \rangle \in HAMPATH \text{ iff } f(\langle G, s, t \rangle) \in TSP,$$

where $f(\langle G, s, t \rangle) = \langle G', s, t, m \rangle$ with G' the graph G with a weight of 1 on all of its edges and m the number of nodes in G. Clearly, the reduction runs in polynomial time. We verify the reduction condition by first observing that a Hamiltonian path gives rise to a minimal traveling salesman circuit by the virtue that all Hamiltonian paths in G' have the same cost. The converse also holds, if we have a traveling salesman circuit this implies that we have a Hamiltonian path. □