# Chapter Five: Nondeterministic Finite Automata

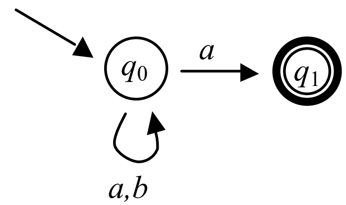
# From DFA to NFA

- A DFA has exactly one transition from every state on every symbol in the alphabet.
- By relaxing this requirement we get a related but more flexible kind of automaton: the nondeterministic finite automaton or NFA.

# Outline

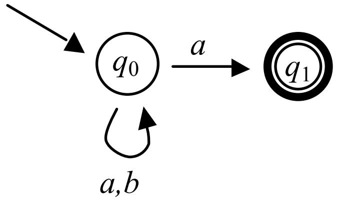
- 5.1 Relaxing a Requirement
- 5.2 Spontaneous Transitions
- 5.3 Nondeterminism
- 5.4 The 5-Tuple for an NFA
- 5.5 The Language Accepted by an NFA

# Not A DFA



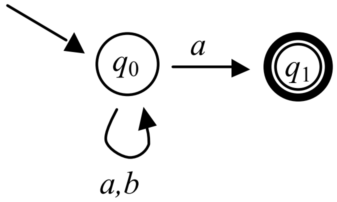

- Does not have exactly one transition from every state on every symbol:
  - Two transitions from $q_0$ on *a*
  - No transition from $q_1$ (on either *a* or *b*)
- Though not a DFA, this can be taken as defining a language, in a slightly different way

# Possible Sequences of Moves




- We'll consider all possible sequences of moves the machine might make for a given string
- For example, on the string *aa* there are three:
  - From $q_0$ to $q_0$ to $q_0$, rejecting
  - From $q_0$ to $q_0$ to $q_1$, accepting
  - From $q_0$ to $q_1$, getting stuck on the last *a*
- Our convention for this new kind of machine: a string is in $L(M)$ if there is *at least one* accepting sequence

# Nondeterministic Finite Automaton (NFA)



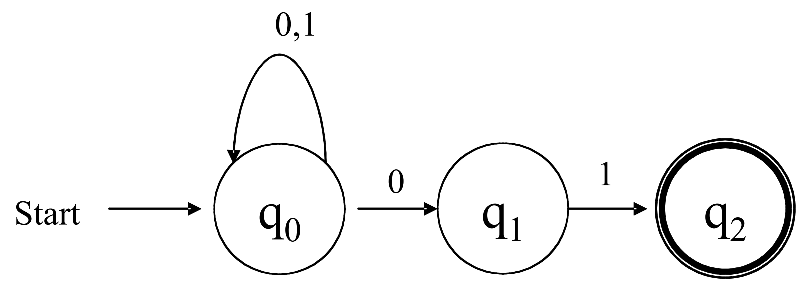

- $L(M)$ = the set of strings that have *at least one* accepting sequence
- In the example above, $L(M) = \{xa \mid x \in \{a,b\}^*\}$
- A DFA is a special case of an NFA:
  - An NFA that happens to be deterministic: there is exactly one transition from every state on every symbol
  - So there is exactly one possible sequence for every string

# Nondeterminism

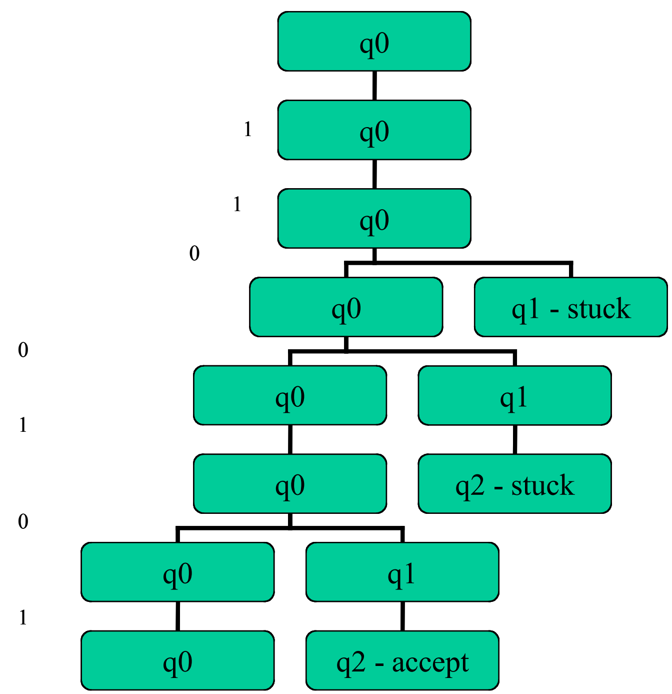
- The essence of nondeterminism:
  - For a given input there can be more than one legal sequence of steps
  - The input is in the language if at least one of the legal sequences says so
- We can achieve the same result by computing all legal sequences in parallel and then deterministically search the legal sequences that accept the input, but…
- ...this nondeterminism does not directly correspond to anything in physical computer systems
- In spite of that, NFAs have many practical applications

# NFA Example

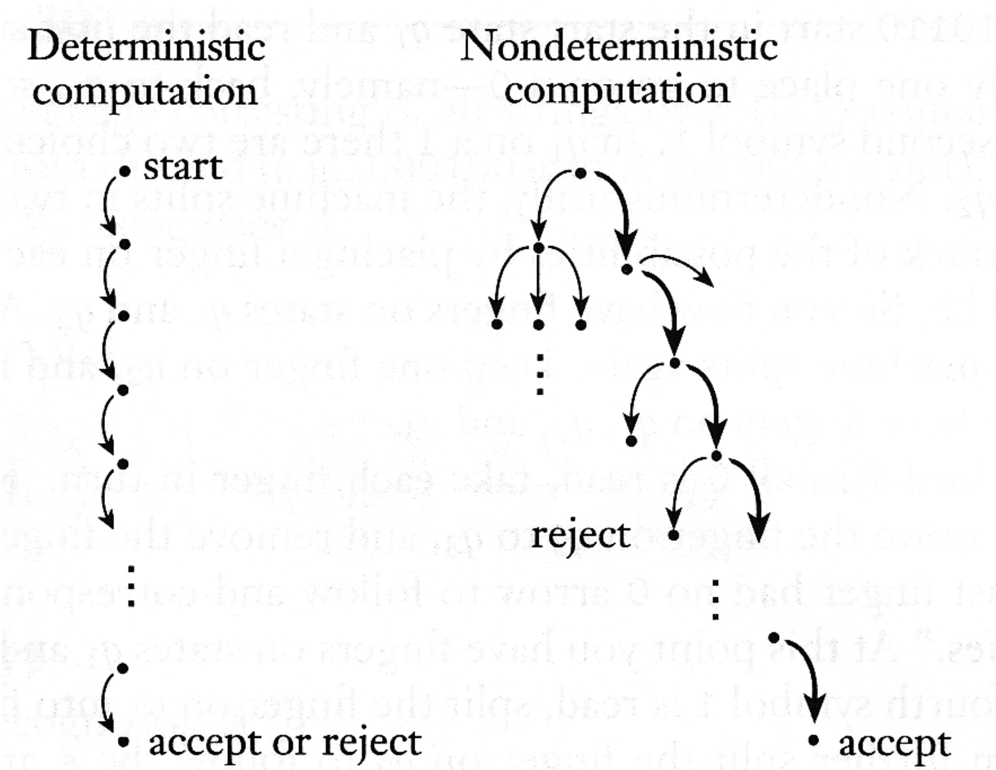
- This NFA accepts only those strings that end in 01



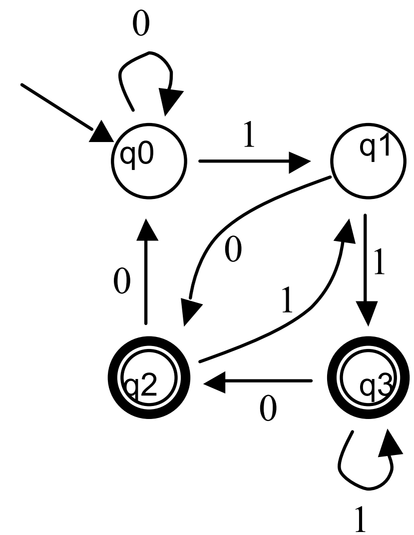

- Running in "parallel threads" for string 1100101



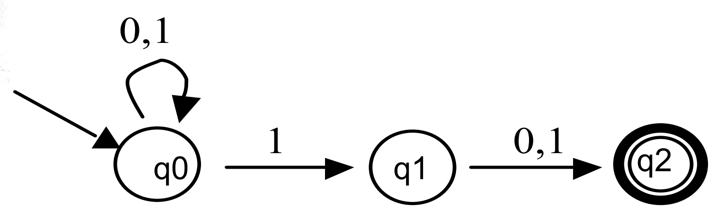
# Nondeterminism




DFA:




NFA:




Now consider string: 0110

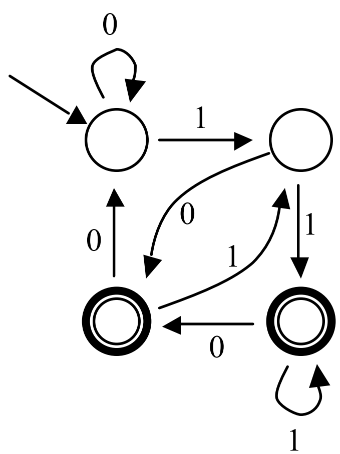
# DFAs and NFAs

- DFAs and NFAs both define languages
- DFAs do it by giving a simple computational procedure for deciding language membership:
  - Start in the start state
  - Make one transition on each symbol in the string
  - See if the final state is accepting
- NFAs do it by considering all possible transitions *in parallel*.

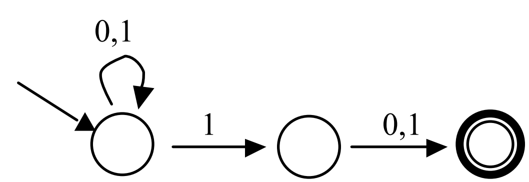
# NFA Advantage

- An NFA for a language can be smaller and easier to construct than a DFA
- Let L={x $\in$ {0,1}*|where x is a string whose next-to-last symbol is 1}
- Construct both a DFA and NFA for recognizing L.

DFA:




NFA:



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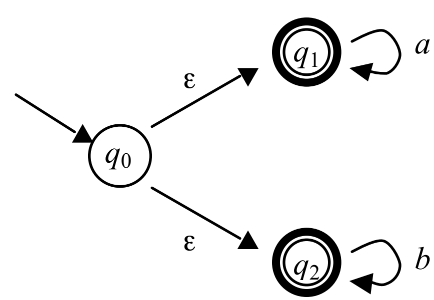
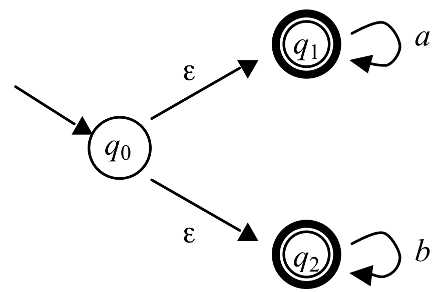
# Spontaneous Transitions

- An NFA can make a state transition spontaneously, without consuming an input symbol
- Shown as an arrow labeled with ε
- For example, $\{a\}^* \cup \{b\}^*$:

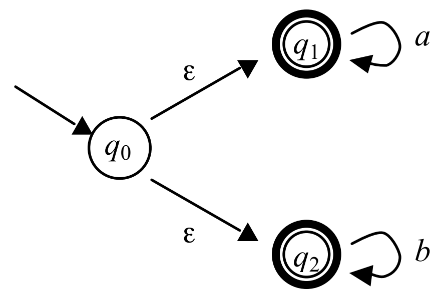


# ε-Transitions To Accepting States



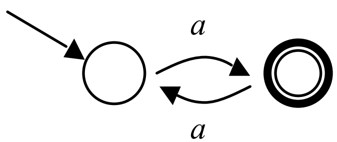

- An ε-transition can be made at any time
- For example, there are three sequences on the empty string
  - No moves, ending in $q_0$, rejecting
  - From $q_0$ to $q_1$, accepting
  - From $q_0$ to $q_2$, accepting
- Any state with an ε-transition to an accepting state ends up working like an accepting state too

# ε-transitions For NFA Combining

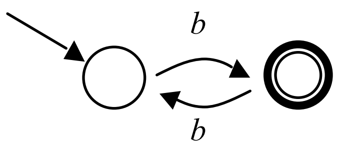



- ε-transitions are useful for combining smaller automata into larger ones
- This machine is combines a machine for {*a*}* and a machine for {*b*}*
- It uses an ε-transition at the start to achieve the union of the two languages

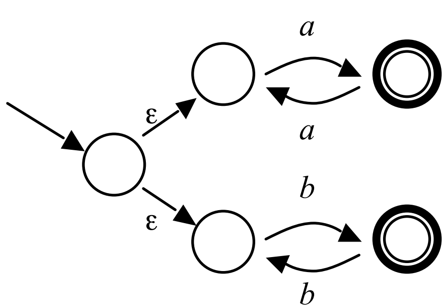
# Revisiting Union




$A = \{a^n \mid n \text{ is odd}\}$

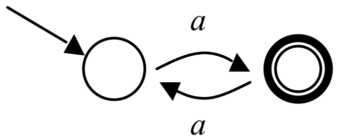



$B = \{b^n \mid n \text{ is odd}\}$

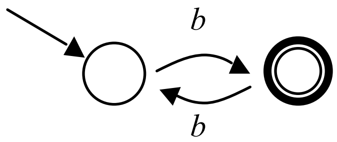



$A \cup B$

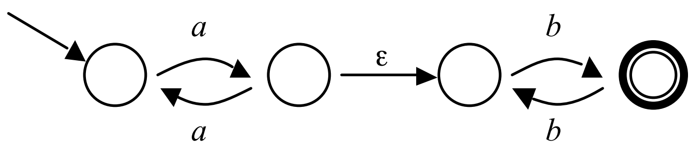
# Concatenation




$A = \{a^n \mid n \text{ is odd}\}$



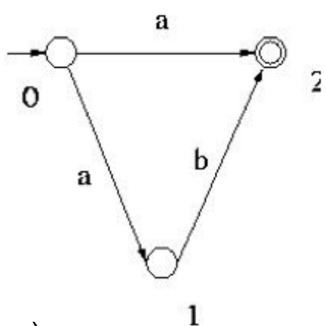

$B = \{b^n \mid n \text{ is odd}\}$



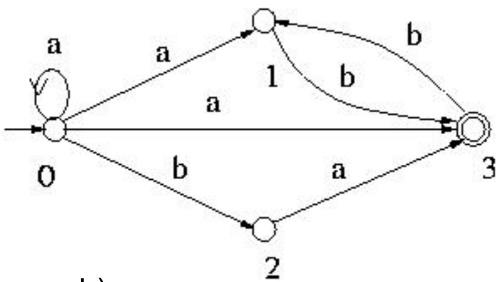

$\{xy \mid x \in A \text{ and } y \in B\}$

# Some Exercises

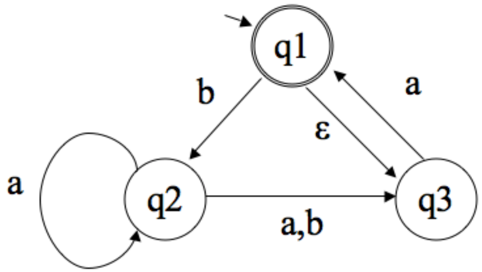
What is the language accepted by the following NFAs?




a)




b)




c)

# More Exercises

- Let Σ = {a, b, c}. Give an NFA M that accepts:

  L = {x | x is in $\Sigma^*$ and x contains ab}

# One More Exercise

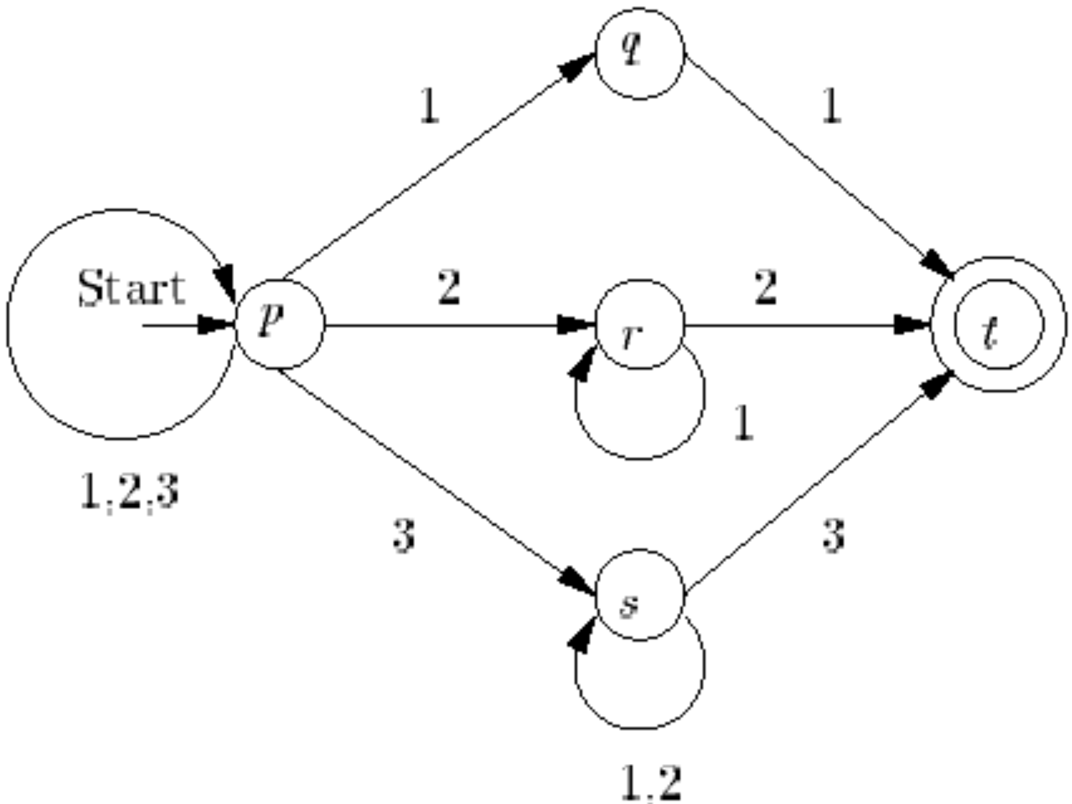
- Let $\Sigma = \{a, b\}$. Give an NFA M that accepts:

  $L = \{x \mid x \text{ is in } \Sigma^* \text{ and the third to the last symbol in } x \text{ is b}\}$

# NFA Exercise

- Construct an NFA that will accept strings over alphabet {1, 2, 3} such that the last symbol appears at least twice, but without any intervening higher symbol, in between:
  - e.g., 11, 2112, 123113, 3212113, etc.
- Trick: use start state to mean "I guess I haven't seen the symbol that matches the ending symbol yet." Use three other states to represent a guess that the matching symbol has been seen, and remembers what that symbol is.
- Spoiler Alert: answer on the next slide!

# NFA Exercise (answer)



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# Powerset

- If $S$ is a set, the *powerset* of $S$ is the set of all subsets of S:

$$P(S) = \{R \mid R \subseteq S\}$$

- This always includes the empty set and $S$ itself
- For example,

P({1,2,3}) = {{}, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}}

# The 5-Tuple

An NFA $M$ is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where:

$Q$ is the finite set of states

$\Sigma$ is the alphabet (that is, a finite set of symbols)

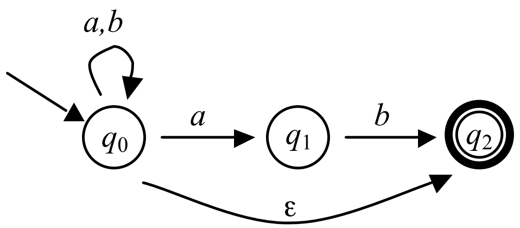
$\delta \in (Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow P(Q)$ is the transition function

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accepting states

- The only change from a DFA is the transition function $\delta$
- $\delta$ takes two inputs:
  - A state from Q (the current state)
  - A symbol from $\Sigma \cup \{\varepsilon\}$ (the next input, or $\varepsilon$ for an $\varepsilon$-transition)
- $\delta$ produces one output:
  - A subset of Q (the set of possible next states - since multiple transitions can happen in parallel!)

# Example:




- Formally, $M = (Q, \Sigma, \delta, q_0, F)$, where
  - $Q = \{q_0,q_1,q_2\}$
  - $\Sigma = \{a,b\}$ (we assume: it must contain at least *a* and *b*)
  - $F = \{q_2\}$
  - $\delta(q_0,a) = \{q_0,q_1\}$, $\delta(q_0,b) = \{q_0\}$, $\delta(q_0,\varepsilon) = \{q_2\}$,
    $\delta(q_1,a) = \{\}$, $\delta(q_1,b) = \{q_2\}$, $\delta(q_1,\varepsilon) = \{\}$
    $\delta(q_2,a) = \{\}$, $\delta(q_2,b) = \{\}$, $\delta(q_2,\varepsilon) = \{\}$
- The language defined is $\{a,b\}^*$

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# The $\delta^*$ Function

- The $\delta$ function gives 1-symbol moves
- We'll define $\delta^*$ so it gives whole-string results (by applying zero or more $\delta$ moves)
- For DFAs, we used this recursive definition
  - $\delta^*(q,\varepsilon) = q$
  - $\delta^*(q,xa) = \delta(\delta^*(q,x),a)$
- The intuition is similar for NFAs taking parallel transitions into account, but the $\varepsilon$-transitions add some technical difficulties

# NFA IDs

- An *instantaneous description* (ID) is a description of a point in an NFA's execution
- It is a pair $(q,x)$ where
  - $q \in Q$ is the current state
  - $x \in \Sigma^*$ is the *unread* part of the input
- Initially, an NFA processing a string $x$ has the ID $(q_0,x)$
- An accepting sequence of moves ends in an ID $(f,\varepsilon)$ for some accepting state $f \in F$

# The One-Move Relation On IDs

- We write

  $$I \mapsto J$$

  if $I$ is an ID and $J$ is an ID that could follow from $I$ after one move of the NFA
- That is, for any string $x \in \Sigma^*$ and any a $\in \Sigma$ or a = $\varepsilon$,

$$(q, ax) \mapsto (r, x) \text{ if and only if } r \in \delta(q, a)$$

# The Zero-Or-More-Move Relation

- We write

  $$I \mapsto^* J$$

  if there is a sequence of zero or more moves that starts with *I* and ends with *J:*

  $$I \mapsto \cdots \mapsto J$$

- Because it allows zero moves, it is a *reflexive* relation: for all IDs *I*,

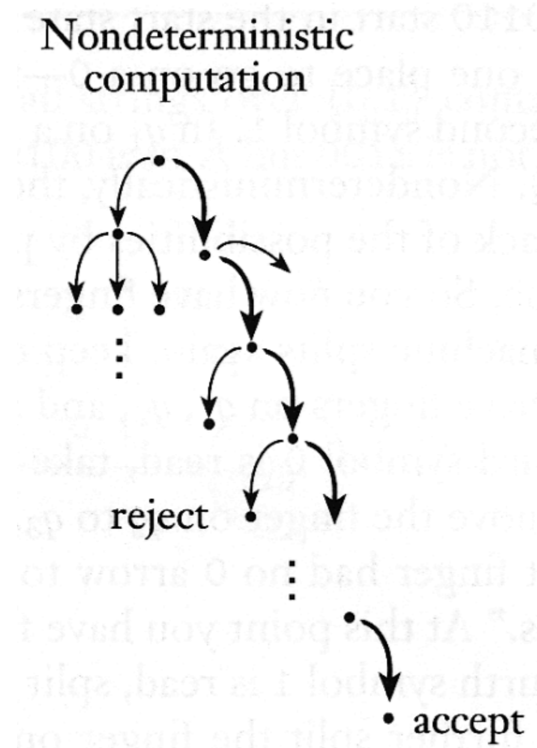
  $$I \mapsto^* I$$

# The δ* Function

- Now we can define the δ* function for NFAs:

$$\delta^*(q,x) = \left\{ r \,\middle|\, (q,x) \mapsto^* (r,\varepsilon) \right\}$$

- Intuitively, δ*($q$,$x$) is the set of all states the NFA might be in after starting in state $q$ and reading $x$



# $M$ Accepts $x$

- Now $\delta^*(q,x)$ is the set of states $M$ may end in, starting from state $q$ and reading all of string $x$
- So $\delta^*(q_0,x)$ tells us whether $M$ accepts $x$ by computing all possible states by executing all possible transitions in parallel on the string $x$:

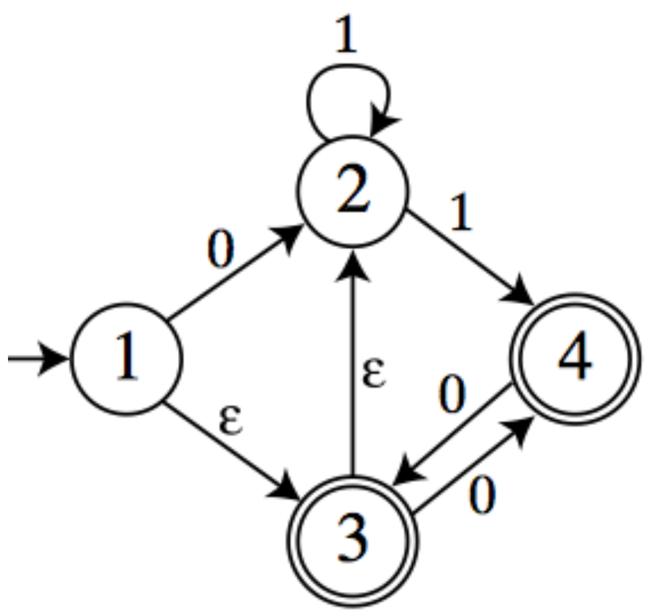
A string $x \in \Sigma^*$ is accepted by an NFA $M = (Q, \Sigma, \delta, q_0, F)$ if and only if the set $\delta^*(q_0, x)$ contains at least one element of $F$.

# The Language An NFA Defines

For any NFA $M = (Q, \Sigma, \delta, q_0, F)$, $L(M)$ denotes the language accepted by $M$, which is

$$L(M) = \{x \in \Sigma^* \mid \delta^*(q_0, x) \cap F \neq \{\}\}.$$

# Exercise




- Compute the results of the following transitions:
  - $\delta^*(q1,\varepsilon)$
  - $\delta^*(q1,0110)$

# Assignment

- Assignment #3 – see website