THE MISSING LINK AND THE TOP-SPIN

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Abstract

The missing link and the top spin are two puzzles closely related to Loyd's 15-puzzle. Basic group-theoretic machinery required to solve these puzzles is presented along with their complete solutions. New variations of these puzzles are suggested and analyzed. Some observations are also made regarding the efficiency of presented solutions.

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1. Introduction

Sliding piece puzzles, the best known example of which is Sam Loyd's 15-puzzle, have been designed and marketed in numerous forms for more than a century. Hordern's book [Hor86] is perhaps the most comprehensive treatment of such puzzles. Hordern presents hundreds of such puzzles with their solutions. This book does not give a mathematical treatment of its subject, however. A brief appendix by D. Singmaster provides a summary of some group-theoretic results underlying these puzzles. A mathematical treatment of such puzzles was carried out by R. Wilson [Wil74] and extended by D. Kornhauser et al [Kor84]. We will present a group-theoretic analysis of two related puzzles marketed under the names "missing link" and "top spin" and also introduce many new variations of them.

The sliding piece puzzles can be viewed as being played on an undirected graph $G = < V, E >$ in which some vertices have pebbles. Throughout we will assume that $V = \{1, 2, ..., n\}$. The only rule of the game is to slide a pebble from a vertex to an adjacent vertex provided the latter is unoccupied. It is assumed that at any time at least one vertex is not occupied. The game consists of a sequence of such moves so that in the final configuration the set of occupied vertices $V' \subseteq V$ is the same as in the beginning. Assume

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that $|V'| = k$ and that $V' = \{1, 2, ..., k\}$. The puzzle consists of several instances each defined by a pair of permutations (s, f) where $s,\ f\ : V' \rightarrow \{1, 2, ..., k\}$. s is the starting configuration in which vertex $x \in V'$ contains the pebble $s(x)$. f is the "target" or "goal" permutation. If the instance (s, f) has a solution we say that the permutation π which transforms s into f is *realizable*. The set of realizable permutations is easily seen to form a group and therefore a subgroup of the symmetric group on k elements. Thus powerful group-theoretic techniques can be used to solve the sliding puzzles. Sam Loyd's original puzzle corresponds to the graph shown in Figure 1.

Wilson [Wil74] was the first to generalize Loyd's 15-puzzle to an arbitrary (undirected) graph but he only considered $k = n - 1$ (i.e. exactly one vertex is unoccupied at any time) as in 15-puzzle. He showed that if G is biconnected and has at least nine vertices then the underlying group is either the alternating group (if G is bipartite) or the symmetric group (if it is not bipartite). Kornhauser et al. [Kor84] further generalized Loyd-Wilson's puzzle by allowing any $k\ \leq\ n - 1$ and provided a characterization of the solvable instances. The puzzles we study here (the missing link and the top spin) are different from the sliding piece puzzles in that sliding moves are augmented or replaced by moves such as rotation and/or reversal. A more formal treatment of group-theoretic puzzles including efficiency and optimality issues is presented in [Rav94].

2. THE MISSING LINK

Missing link is a product of the Ideal Toy Corporation, the makers of Rubik's magic cube. The puzzle consists of four small cubes joined together to form a $4 \times 1 \times 1$ shape. Three of the four faces of 4×1 rectangular shape contains an interlocked 3-chain; the fourth one contains an interlocked 2-chain and an empty slot. Figure 2 shows the visible part of the missing link. Each chain on a face is marked by a distinct color: red, green, yellow and white. White is the special color which contains the 2-chain. The clever design of the puzzle is so as to allow to stay as a single piece but nevertheless allow the following two types of moves: (i) a 1×1 square (call it a *link*) can be slid to an unoccupied horizontal neighboring slot and (ii) the first or the last cube can be rotated by 90^0, 180^0 or 270^0. The goal of the puzzle is to bring it from an arbitrary position to the 'goal' position such as the one in Figure 2. This final position is not unique since the puzzle does not specify the colors on the faces. For example, interchanging the chains in the front and top faces (if it is possible) we will get a different solution. It is not difficult to recognize almost immediately (almost it took the author some time) the close relationship between the missing link and 15-puzzle. Both have 16 vertices and 15 pieces and the moves are quite similar.

Since we want to relate this puzzle to the 15-puzzle, it is convenient to use numbers in the place of links. We will thus number the links 1 through 15 (rather arbitrarily) as follows: Red links (from left to right in the chain) will be numbered 1 to 4, green from 5 to 8, yellow from 9 to 12 and white from 13 to 15. (This actually makes the puzzle slightly more difficult to solve since the original puzzle did not enforce relative position of any chain with respect to any other; further the middle pairs of red, green and yellow links are identical and thus their relative positions are not enforced either.) Next we cut and flatten the faces so that the puzzle exactly looks like 15-puzzle. The horizontal sliding rule is in fact exactly like in 15-puzzle, but the vertical move is different. A link is not allowed to slide vertically. Instead, the first or the fourth column links can be cyclically rotated up or down. See Figure 3 and 4 for examples of permitted moves. We will also assume that at the beginning and the end of the puzzle, the right bottom corner slot is empty. Then the pieces in the other 15 slots define a permutation of $[1, 15] = \{1, 2, ..., 15\}$. Again it is not difficult to see that the set of solvable permutations is a group. Two basic questions we would like to address are: (i) Characterize this group and (ii) Provide an algorithm to realize any permutation in the group.

First we will briefly discuss the above two questions for the 15-puzzle. This analysis was provided by W. W. Johnson [Jon79] and W. E. Story [Sto79] more than one hundred years ago. Let G_1 be the set of solvable permutations. We will show that G_1 is the alternating group A_{15} of the set of even permutations on [1, 15].

Lemma 1. *G_1 is a subgroup of A_{15}.*

Proof. Assign a fictitious label 16 to the empty slot. Now we can view a board position at any intermediate step as a permutation of [1, 16]. Of course at the beginning and end 16 maps to itself and so such positions are essentially permutations of [1, 15]. Each move is a transposition of the form (16 x). Further each move moves 16 to an adjacent vertex. Since 16 is required to return to its original vertex at the end, each of its north (west) move must have a corresponding south (east) move and so if π is the permutation which maps the starting position to the final position, it is a product of an even number of transpositions. Thus G_1 is a subgroup of A_{15}. •

To complete the proof we need to show the converse: every even permutation is realizable. The next lemma presents a standard technique for showing this. Let S_n be the symmetric group of all permutations on $[1, n]$. A group $G \subseteq S_n$ is said to be 2-transitive if for any $x_1, x_2, x_3, x_4 \in [1, n]$ ($x_1 \neq x_2$, $x_3 \neq x_4$), there is a $\pi \in G$ such that $\pi(x_1) = x_3$ and

$\pi(x_2) = x_4$. The next lemma presents a useful technique for showing that a certain group presented by a set of generators is the alternating or the symmetric group. Throughout, we assume that the permutations enclosed between "(" and ")" are in cycle notation and that the permutations are composed from left to right.

Lemma 2. ([Kor84]) Let $G \subseteq S_n$ be a 2-transitive group which contains a fixed 3-cycle, say (1 2 3). Then G contains all the even permutations, i.e., G includes A_n.

Proof. We will show that G includes all 3-cycles. Let $a, b, c \in [1, n]$. To show that $(abc) \in G$ let f be a permutation such that $f(1) = a$, $f(2) = b$. Clearly $f^{-1}(123)f$ is a 3- cycle of the form $(a\ b\ d) \in G$ for some d. If $d = c$ then we are done. Otherwise by a similar conjugation we can show that $(b\ c\ e) \in G$ for some e. If $d = e$ then using $A = (a\ b\ d)$ and $B = (b\ c\ d)$ we see that $BA = (a\ b\ c)$ is in G. If $d \neq e$ then we proceed as follows: $(b\ c\ e)^{-1}(a\ b\ d)(b\ c\ e) = (a\ c\ d)$ is in G. Using $(a\ c\ d)$ and $(a\ b\ d)$, we get $(a\ b\ c)$ as: $(a\ b\ c) = (a\ b\ d)(a\ c\ d)^2$. Finally note that any $\sigma \in A_n$ is a product of 3-cycles as shown below. Since σ is an even permutation it can be expressed as a product of an even number of transpositions $\sigma = (a_{2k}\ b_{2k})\ (a_{2k-1}\ b_{2k-1})\ ...\ (a_1\ b_1)$. We can replace a pair of adjacent transpositions by a 3-cycle or a product of two 3-cycles using the following rules: (i) $(a\ b)(a\ c) = (a\ b\ c)$, (ii) $(a\ b)(c\ d) = (a\ b\ d)(c\ d\ a)$. This completes the proof of Lemma 2. •

We can now prove the main result about 15-puzzle.

Theorem 1. G_1 is the alternating group A_{15}.

Proof. In view of Lemma 2, it is sufficient to show that G_1 is 2-transitive and that it contains a 3-cycle. The latter is trivial to prove: The move sequence 12-11-15-12 (numbers denote the moving pieces) produces the 3-cycle (11 15 12). To show 2-transitivity note that it is sufficient to show that for any distinct $a, b \in [1, 15]$ there is a $p \in G_1$ such that $p(1) = a$ and $p(2) = b$, as argued below: Let $a, b, c, d \in G_1$. Suppose we want to find p such that $p(a) = c$, $p(b) = d$. Let q be such that $q(1) = a$, $q(2) = b$ and r be such that $r(1) = c$ and $r(2) = d$. Then p can be chosen as rq^{-1}.) We will now describe such a move. First note that a 15-cycle (call it c) shown in Figure 5(a) by the closed walk traversed by the empty slot (starting and ending in the southeast corner) is in G_1. Clearly c^t for some t will move a into position 1. (Here c^t denotes the composition of c t-times.) Let c_1 and c_2 be permutations realized by the sequence of moves shown in Figure 5(b) and (c). c_1 is a 13-cycle which fixes only 1 and 5. Thus if $c^t(5)$ is not b, then the move sequence c^t

has left b in a vertex which is included in the cycle c_1 and so c^t followed by c_1^u will move b to vertex 2 without disturbing a. If $c^t(5)$ is b then we can use the cycle c_2 instead. This completes the proof. •

Analyzing the missing link requires little more group theory than that used in the proof of Theorem 1 but the details are more complicated. Let G_2 denote the subgroup of S_{15} associated with the missing link puzzle. The next theorem characterizes G_2.

Theorem 2. G_2 is the symmetric group S_{15}.

Proof. It is enough to show that G_2 includes all the even permutations and at least one odd permutation for the claim then follows from Lagrange's theorem which states that the order of a group should be divisible by the order of any subgroup. The latter claim is trivial since the rotation of the first or the last column (when it does not contain the empty slot) induces a 4-cycle which is an odd permutation. The rest of the proof involves showing that G_2 includes all the even permutations and we do this by showing that G_2 is 2-transitive and that it includes a 3-cycle.

Figure 6(a) through 6(i) shows a sequence of moves which induces the the 3-cycle (11 15 12). To show 2-transitivity, we proceed as in Theorem 1. It suffices to show that for any distinct $i, j \in \{1, 2, ..., 15\}$ there is a permutation $p \in G_2$ such that $p(1) = i$ and $p(2) = j$. In Figures 7, 8(a), 8(b) and 8(c) are described a sequence of moves in notation similar to Figure 5. It can be seen that the permutation produced by the sequence of moves shown in Figure 7 is $\sigma = (1\ 2\ 3\ 8\ 15\ 14\ 13\ 5\ 10\ 11\ 12\ 4\ 7\ 6\ 9)$, a 15-cycle and that the permutations produced by the move sequences in 8(a), 8(b) and 8(c) are $\sigma_1 = (1\ 13\ 14\ 15\ 4\ 8\ 12\ 11\ 10\ 9\ 5)$, $\sigma_2 = (1\ 9)(4\ 12)(5\ 13\ 14\ 15\ 8\ 7\ 6)$ and $\sigma_3 = (\ 1\ 5\ 9\ 13\ 14\ 15\ 12\ 8\ 4\ 3\ 2)$, respectively. It thus follows that $\tau = (\sigma_1\sigma_3)^{-2} = (2\ 4\ 11\ 5\ 3\ 12\ 10)$ and $\rho = \sigma_2^2 = (5\ 14\ 8\ 6\ 13\ 15\ 7)$ are in G_2. We are now ready to exhibit p for a given i and j as required in the beginning of this paragraph.

Since σ is a 15-cycle, we can find a suitable k such that $\sigma^k(1) = i$. Note that $\tau\rho$ is a 13-cycle $(5\ 3\ 12\ 10\ 2\ 4\ 11\ 14\ 8\ 6\ 13\ 15\ 7) \in G_2$. Note that 9 and 1 are the only elements fixed by $\tau\rho$. Thus if $\sigma^k(9)$ is not j then σ^k followed by a suitable power of $\tau\rho$ will move j to vertex 2 without disturbing i which was moved to 1 by σ^k. If $\sigma^k(9) = j$ then we need another cyclic permutation in G_2 which includes 9 but not 1. Such a permutation is easy to find, e.g. $\sigma_4 = (2\ 3\ 8\ 7\ 6\ 13\ 14\ 15\ 11\ 10\ 9)$ induced by the sequence of moves illustrated in Figure 9. This completes the proof. •

The above proof can be simplified to some extent. However we have chosen to present this because it provides all the necessary machinery to analyze a variation of the missing link puzzle in which the rotating move in the missing link is modified as follows: The first or the last column can be rotated only if one of the vertices is unoccupied. The other rules are exactly the same as in the missing link. This variation was suggested during a talk given by the author in a weekly combinatorics seminar at the University of Rhode Island. It turns out that the underlying group (let us call it G_3) is the alternating group and thus any even permutation can be realized. This fact is somewhat surprising since physical experimentation with the puzzle gives a feeling that the options have become quite limited and hence the number of realizable permutations is perhaps quite small. It took us sometime to identify a 3-cycle.

Theorem 3. G_3 is the alternating group A_{15}.

Proof. We need to show the following: (i) G_3 contains a 3-cycle, (ii) G_3 is 2-transitive and (iii) G_3 does not contain any odd permutations. A proof of (ii) can be taken verbatim from the proof of Theorem 2 since it never used the 4-cycle move of the missing link. A proof of (iii) is very similar to that of Lemma 1 and we leave out the details. Note, however, that the claim about 3-cycle in the proof of Theorem 2 does not carry over. We will now show (i). The claim in Theorem 2 that τ and ρ are in G_2 carry over to G_3 as well. Their commutator $[\tau, \rho] = \tau\rho\tau^{-1}\rho^{-1}$ can be seen to be the 3-cycle (5 7 11). •

One may wonder whether the algorithm presented in the proof of Theorem 2 (or Theorm 3) is amenable for human use in physically solving the missing link puzzle. Probably not! It seems that the typical trial and error approach adopted by most human solvers will lead them (as in 15-puzzle) to solve it row by row. This approach is easier in solving the 15-puzzle in which the top rows need not be disturbed once they are done. Since the missing link lacks this property, it is slightly more complicated to use the row-by-row approach; it is nevertheless the most promising method for manual solution of the missing link. However the row-by-row algorithm becomes complex if we want to present it in complete detail (as required of a mathematical proof). Also we do not know a row-or-row strategy for the modified missing link and hence this puzzle may provide hours of fun (or frustration) to puzzle buffs.

Finally we would like to discuss the issue of efficiency of the solution of the missing link. Two (almost conflicting) notions of efficiency are relevant here: First one is the number of moves required to solve the puzzle (a move being a sliding move or a rotation).

Thus a solution which requires fewer moves is considered more efficient and an optimal algorithm is the one which makes the fewest moves for any starting and goal configurations. Another notion of efficiency arises in the context of solving such puzzles using a computer program and it denotes the time taken by such a computer program to produce a solution. Typically these two efficiencies are "complementary" in the following sense. If the program does not exhaustively search for all possible move sequences, it can produce a solution fast but it may not be the shortest or even close to being shortest. Conversely an exhaustive search will find the shortest solution but may be prohibitive to implement. Of course it would be desirable if we can achieve the better of both, namely an algorithm which finds the shortest move sequence efficiently, i.e., without an exhaustive search. A widely accepted notion of efficiency in theoretical computer science circles is that the running time be bounded by a polynomial in the size of the input. (Here the input is the graph G and the starting and the goal configurations.) Goldreich [Gol84] shows that an optimal and efficient general-purpose algorithm is not likely to exist for all the sliding piece puzzles. More technically he shows that the following decision problem is NP-complete: "Given a graph G, starting and final configurations s and t, and an integer k, is there a sequence of k or fewer sliding moves which will transform s into t?" However Goldreich's result does not preclude the possibility that efficient optimal strategies exist for special classes of graphs. In particular we do not know if such an algorithm exists for the $\sqrt{n} \times \sqrt{n}$ generalization of 15-puzzle. In [Rav94] efficient optimal algorithms are exhibited for some classes of graphs including (2,n)-bipartite graph, wheel graphs and open-wheel graphs.

A related problem of combinatorial flavor is as follows. For a puzzle P let $d(P)$ denote the minimum number of moves required to transform a configuration s into a configuration t, maximized over all pairs (s, t) such that t is reachable from t. (Here a sliding, rotating or a reversing operation is counted as a move.) The problem is to estimate $d(P)$ for various puzzles. $d(P)$ is known as the diameter of the group and has been studied before (e.g. [Dri83], [Jer85] etc). Estimating the exact value of $d(P)$ is quite difficult even for simple puzzles without the help of a computer. Some exceptions can be found in [Jer85], [Rav94]. A simpler problem is to find asymptotic estimates for a class of graphs as the number of vertices $n \to \infty$. A main result shown in [Kor84] is such an asymptotic estimate: for a sliding piece puzzle over any graph G, the diameter is bounded by $O(n^3)$ where n is the number of vertices in G. This bound is tight for some classes of graphs but not in general. For example, it is not difficult to show that the diameter of the $\sqrt{n} \times \sqrt{n}$ generalization of 15-puzzle is $\Theta(n^{1.5})$. We can generalize the missing link in the same way to $\sqrt{n} \times \sqrt{n}$ graph in which horizontal or row-wise sliding moves and the rotation of the first and the last column are allowed. It is not difficult to extend Theorem 2 and show that the group which underlies this puzzle is the symmetric group S_{n-1}. The diameter of this group can be shown to be upper-bounded by $O(n^2)$ [Rav94] but we do not know if this bound is tight.

3. TOP SPIN

Top-Spin allows rotating and reversing moves but not sliding moves. As shown in Figure 10, it consists of a circular arrangement of 20 pebbles numbered 1 through 20. The pieces can be slid along the loop clockwise or counterclockwise. There is also a platform on this loop which can hold exactly four contiguous pebbles. The platform can be rotated by 180^0 and its effect obviously is to reverse the four pebbles on it. These are the only allowed moves. The original puzzle comes with the pieces ordered 2 1 3 4 5 ... 20. The object is to interchange 1 and 2. However, as with the missing link, we will solve the more general problem of characterizing the underlying group which we denote $G_{4,20}$. We will then present some variations and generalizations of top spin.

Theorem 4. $G_{4,20}$ is the symmetric group S_{20}.

Proof. We will show how to switch a pair of adjacent pebbles. It is easy to see that any arbitrary permutation can be realized using the ability to switch an adjacent pair of elements. A move sequence for top spin can be represented by a sequence such as 3, R, 2, R, -1. R denotes the reversal move and the integers denote the amount by which the pebbles are rotated, positive numbers denoting clockwise rotation, negative numbers counterclockwise rotation. Assume that the starting configuration is the sequence 2 1 3 4 ... 20 (clockwise) with the reversing platform containing 2 1 3 4. It is easy to verify that the sequence of moves 3, R, -1, R, 1, R, -1, R, 1 generates a 5-cycle and repeating this sequence 5 times, we get the desired configuration. •

Since top spin turned out to be very simple, we will introduce some interesting variations which are more difficult to solve. The first one is to replace the circular arrangement by a linear arrangement. More precisely consider 20 slots on a line each containing a pebble numbered 1 through 20. There is a movable platform which can hold 4 pebbles and it can be rotated by 180^o. What permutations can be realized? We leave it as an interesting exercise to show that all and only even permutations can be realized. We will consider our next variation in some detail. This variation is interesting for at least two reasons. The first one is that the variation can be easily incorporated in the design (unlike the variation we considered in the missing link). The other reason is that this puzzle falls into a class of problems which has received a great deal of attention recently in chromosome classification and the variation we are about to study has a natural interpretation in that context. The variation is that the pebbles have an orientation ("up" or "down"). Every time a pebble gets reversed its orientation also changes. Initially all the pebbles are "up" and we want the final configuration to have the same property as well. Call this the oriented-top-spin. In effect, oriented version has an additional requirement that each pebble be reversed an even number of times. A simple way to physically realize this condition is to make the pebbles in square shape so that they do not move about their centers (unlike the commercially available puzzle in which the pieces are circular and can rotate about their centers). Is it possible to realize the permutation (1 2) in oriented-top-spin? Answer is "no" as seen from the following simple reasoning. Let $d(i,j)$ be defined as a $\{0,1\}$ valued function (where $i, j \in [1,n]$, $i \neq j$) which denotes the parity of the number of pebbles that lie between i and j (it does not matter if you count clockwise or counterclockwise). Note that d is well defined only because n, the number of pebbles, is 20, an even number. It is easy to see that $d(i,j)$ is invariant under the moves of the puzzle and the claim follows. The next theorem characterizes exactly the set of realizable permutations in oriented-top-spin(4,20).

Theorem 5. A permutation π is realizable if and only if the following conditions hold: (i) $\pi(i)$ for odd values i are all even or all odd (thus the same is true of $\pi(i)$ for even i) and (ii) if π_1 and π_2 be restrictions of π on sets $\{1,3,...,19\}$ and $\{2,4,...,20\}$, then π_1 and π_2 are both even or both odd.

Proof. We leave out the necessary part which is not difficult to show. To show that the above conditions are sufficient we first show that the permutations (a) (1 2)(3 4) and (b) (1 3 5) belong to oriented-top-spin(4,20). We do this by exhibiting a sequence of useful maneuvers or processes and then make repeated use of these processes. In the following, such a process are assigned a letter name such as A, B etc. In describing some of the latter processes, we may either use the basic moves (of rotation or 4-reversal) or an earlier process. We also explain in parenthesis how each step was realized. We use i' etc. to denote the pebbles in "down" orientation. Note that the permutations presented are in two-line notation with the top-line suppressed.

process A: start: 1 2 3 ... 20 finish: 2' 3' 4' 5' 1 6 7 ... 20

We skip the details of this process because this process the steps are identical to the way a 5-cycle was realized in Theorem 4.

process B: start: 1 2 3 ... 20 finish: 2 1 3' 4' 5' ... 20'

1 2 3 4 5 6 ... 20

2' 3' 4' 5' 1 6 7 8 9 ... 20 (A applied to 1 2 3 4 5)

2' 3' 4' 5' 6' 7' 8' 9' 1 ... 20 (A applied to 1 6 7 8 9)

2' 3' ... 9' 10' 11' 12' 13' 1 ... 20 (A applied to 1 10 11 12 13)

3' ... 13' 14' 15' 16' 17' 1 18 19 20 2' (A applied to 1 14 15 16 17)

2 1 3' 4' 5' 6' ... 20' (A applied to 1 18 19 20 2')

process C: the permutation (1 2)(3 4)

1 2 3 4 ... 19 20

2 1 3' 4' 5' ... 19' 20' (B applied to 1 2)

2' 1' 4' 3' 5 6 7 ... 19 20 (B applied to 3' 4')

3 4 1 2 5 6 7 ... 19 20 (reversing 2' 1' 4' 3')

Finally we show how to realize (1 3 5):

1 2 3 4 5 6 ... 19 20

3 4 1 2 5 6 ... 19 20 (C applied to 1 2 3 4)

3 2 5 4 1 6 ... 19 20 (C applied to 4 1 2 5)

Similarly we can show that (2 4 6) is realizable. It is easy to see that the generator set $\{(1\ 3\ 5), (3\ 5\ 7), \ldots, (17\ 19\ 1), (19\ 1\ 3)\}$ generates the alternating group on $\{1, 3, ..., 19\}$ so we can generate any even permutation of odd pebbles and any even permutation of even pebbles independently, i.e., without disturbing the odd pebbles. If we want to generate odd permutations in both odd and even components, we apply the process (1 2)(3 4), followed by a permutation whose odd and even components are even permutations. •

Readers familiar with the group theory behind Rubik's cube will realize that the group underlying oriented-top-spin(4,20) is isomorphic to a subgroup of a direct product of wreath products. What about the group $G_{4,19}$ associated with the oriented top spin(4,19)? Interestingly this turns out to be the alternating group A_{19}. An obvious generalization is to consider top spin(r,n) with n pebbles and a reversing platform of length r. [Rav94] presents a complete characterization of the groups $G_{r,n}$ and $G'_{r,n}$ associated with top spin(r,n) and oriented-top-spin(r,n). For example, it turns out that $G_{4,n}$ is the alternating group if n is odd and $n \geq 7$ and the symmetric group if n is even and $n \geq 6$. Determining the diameter of top spin(r,n) seems to be difficult even for fixed small values of r as $n \to \infty$. As is the convention with puzzles involving reversal, only reversing moves are counted. In [Rav94] we compute the exact diameter of top spin(2,n) for all n. Even determining the shortest move sequence to realize specific permutations seems quite hard. For example, consider the original version of top spin(4,20) which is to realize the transpose (1 2). The proof of Theorem 4 indicates that the number of (reversal) moves required for this is at most 20 and in general, it is possible to transpose 1 and 2 in top spin(4,n) with at most n reversals. Computer programs indicate that this bound is tight for small values of n, e.g. $n = 20$. However the best lower bound we can establish for a general n is the following.

Claim. The number of moves required to realize the permutation (1 2) in top spin(4,n) is at least $n/4$ for all even n.

Proof. The proof is a nice application of parity of permutations. Basically we will show that every pebble must have been involved in a reversal at least once. Suppose not. This means that there is a strategy in which 1 and 2 can be switched without ever having some pebble, say i on the reversing platform. We will then show that it is possible to switch 1 and 2 in top spin(4,$n-1$) as well. The problem with the latter claim is that the group $G_{4,n-1}$ cannot contain an odd permutation since both its basic moves induce even permutations, a contradiction! To show the claim made above we just imitate the supposed strategy to switch 1 and 2 in top spin(4,n). The idea is quite simple. Map the pebbles 1 through $i-1$ of top spin(4,$n-1$) to the pebbles 1 through $i-1$ of top spin(4,n) and the pebbles i through $n-1$ of top spin(4,$n-1$) to pebbles $i+1$ through n of top spin(4,n) and carry out exactly the corresponding move. Since i was never reversed, the simulation will not cause any problem. •

We will conclude with another variation of top spin which seems to have some significance in the newly emerging field of theoretical biology. Suppose we replace the circular arrangement of pebbles with a linear arrangement and define a move as reversal of a segment of *any* length. The minimum number of moves needed to transform a permutation s into a permutation t is called the *inversion distance* between s and t. Very recently Kececioglu and Sankoff [Kec93] and Bafna and Pevzner [Baf93] have studied the inversion distance problem and have exactly determined the diameter of the associated group. This connection may provide further motivation to study such puzzles.

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